Dymore User's Manual
Two- and three-dimensional dynamic inflow models

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1 Two-dimensional finite-state generalized dynamic wake theory

The unsteady aerodynamic forces acting on a two-dimensional airfoil were computed by Theodorsen, based on the potential flow approximation and the Kutta condition [1, 2, 3]. A finite state formulation of the same problem was developed by Peters *et al.* [4], and takes the following form

$$A\underline{\dot{\mu}} + \frac{V}{b}\underline{\mu} = \dot{U}_3\underline{c}, \tag{1}$$

where $\underline{\mu}$ is the inflow states array, A and $\underline{c}$ are a given constant matrix and array defined in eqs. (10) and (9), respectively, V is the magnitude of the components of the far flow velocity, $\underline{V}_\infty$, in the plane of the airfoil, *i.e.* $V = \|(I - \bar{a}_1\bar{a}_1^T)\underline{V}_\infty\|$, b the semi-chord length, and U_3 the component of along unit vector $\bar{a}_3$ of the relative velocity vector at the airfoil three-quarter-chord point. The magnitude of the average inflow vector, λ_0, is a linear combination of the inflow states

$$\lambda_0 = \frac{1}{2}\,\underline{b}^T\underline{\mu}, \tag{2}$$

where $\underline{b}$ is a constant array given by eq. (11). This average inflow acts along a unit vector, $\bar{a}_\lambda$, which is the cross product of the far flow velocity $\underline{V}_\infty$ by unit vector $\bar{a}_1$

$$\bar{a}_\lambda = -\frac{\tilde{V}_\infty\bar{a}_1}{\|\tilde{V}_\infty\bar{a}_1\|}, \tag{3}$$

Thus the average inflow vector is

$$\underline{\lambda} = \lambda_0\bar{a}_\lambda. \tag{4}$$

The governing equations of the problem, eqs. (1), form a set of coupled ordinary differential equations; a central difference scheme is used for their solution

$$A\frac{\underline{\mu}_f-\underline{\mu}_i}{\Delta t}+\frac{V}{b}\frac{\underline{\mu}_f+\underline{\mu}_i}{2}=\frac{U_{3f}-U_{3i}}{\Delta t}\ \underline{c}, \tag{5}$$

where the subscripts $(\cdot)_i$ and $(\cdot)_f$ denote quantities computed at the beginning and end of the time step of size Δt. The inflow states at the end of the time step are then readily computed as

$$\underline{\mu}_f=\left(A+\frac{\Delta\tau}{2}I\right)^{-1}\left[(U_{3f}-U_{3i})\ \underline{c}+\left(A-\frac{\Delta\tau}{2}I\right)\underline{\mu}_i\right], \tag{6}$$

where I is identity matrix and $\Delta\tau = V\Delta t/b$ is the non dimensional time step size.

The coefficients of the theory are summarized here for an approximation involving N inflow states. At first, matrix D is defined as

$$D_{mn}=\begin{cases} n/2 & \text{for} \quad n=m+1,\\ -n/2 & \text{for} \quad n=m-1,\\ 0 & \text{for} \quad n\neq m\pm 1. \end{cases} \tag{7}$$

Next, the following two array, $\underline{d}$ and $\underline{c}$, are defined

$$d_n=\begin{cases} 1/2 & \text{for} \quad n=1,\\ 0 & \text{for} \quad n\neq 1, \end{cases} \tag{8}$$

$$c_n=\frac{2}{n}. \tag{9}$$

With these definitions, matrix A becomes

$$A=D+\underline{d}\underline{b}^T+\underline{c}\underline{d}^T+\frac{1}{2}\underline{c}\underline{b}^T. \tag{10}$$

Finally, array $\underline{b}$ is given as

$$b_n=(-1)^{n-1}\frac{(N+n)!}{(N-n)!}\frac{1}{(n!)^2}, \quad b_N=(-1)^{N+1}. \tag{11}$$

2 Three-dimensional finite-state generalized dynamic wake theory

The formulation for the three-dimensional generalized dynamic wake is based on quasi-steady, potential flow with small perturbations [5]. The formulation begins with the continuity and the momentum equations

$$q_{i,j}=0, \tag{12}$$

$$q_{i,0}-V_\infty q_{i,\xi}=-\Phi_{,i}, \tag{13}$$

The q_i are the velocity perturbations, V_∞ the free stream velocity, Φ the pressure and $(\cdot)_\xi$ the non-dimensional time derivative along the free-stream. Pressure has both a convective component, Φ^V, and a perturbation component in the direction of the flow, Φ^A. Separating the two components gives

$$\Phi = \Phi^V + \Phi^A, \tag{14}$$

$$\Phi^A_{,i} = -q_{i,0}, \tag{15}$$

$$\Phi^V_{,i} = V_\infty q_{i,\xi}. \tag{16}$$

Differentiating eq. (15) and using continuity, Laplace's equation is obtained for the pressure

$$\Phi_{,ii} = 0. \tag{17}$$

This equation implies that both the convective and perturbation components of the pressure each satisfy Laplace's equation. Each component can therefore be seen as its own acceleration potential

$$\Phi^A_{,ii} = 0. \tag{18}$$

$$\Phi^V_{,ii} = 0. \tag{19}$$

The boundary conditions for the potential functions are satisfied by matching the pressure distribution along the blades and setting pressure to zero at infinity. Using separation of variables in ellipsoidal coordinates, Laplace's equation can be solved analytically. This particular method of solution provides for a pressure discontinuity across a circular disk. The potential is then written as a Fourier series

$$\Phi(\nu, \eta, \psi, \bar{t}) = \sum_{m=0}^{\infty} \sum_{n=m+1,m+3}^{\infty} P_n^m Q_n^m(i\eta) \left[C_n^m(\bar{t}) \cos m\psi + D_n^m(\bar{t}) \sin m\psi\right]. \tag{20}$$

Here P_n^m and Q_n^m are Legendre functions of the first and second kind and C_n^m and D_n^m unknown coefficients to be determined. The variables ν, η and ψ are the ellipsoidal coordinates. The discontinuity across the disk comes from the fact ν is positive above the disk and negative below the disk.

Integrating the perturbation along the free-stream

$$q_i = -\frac{1}{V_\infty} \int_\xi^\infty \Phi^V_{,i} \, \mathrm{d}\xi \tag{21}$$

$$\Phi^A_{,i} = -q_{i,0}, \tag{22}$$

Now, only considering the z-component of the induced velocity normal to the inflow disk, eqs. (21) and (15) become

$$w = -\frac{1}{V_\infty} \int_0^\infty \frac{\partial \Phi^V}{\partial z} \, \mathrm{d}\xi \tag{23}$$

$$\frac{\mathrm{d}w}{\mathrm{d}\bar{t}} = -\left.\frac{\partial \Phi^A}{\partial z}\right|_{\eta=0}, \tag{24}$$

Viewing the right hand side of eqs. (23) and (24) as linear operators on Φ^A and Φ^V the pressure can be written as

$$\Phi^A + \Phi^V = E_{-1}w^+ + L^{-1}w = \Phi, \tag{25}$$

Since eq. (25) is assumed to be a linear operation, it is also assumed to be invertible. The induced flow is then expressed in a Fourier series in the azimuthal and radial directions

$$w(\bar{r}, \psi, \bar{t}) = \sum_{r=0}^{\infty} \sum_{j=r+1,m+3}^{\infty} \Psi_j^r \left[\alpha_j^r(\bar{t}) \cos r\psi + \beta_j^r(\bar{t}) \sin r\psi \right]. \tag{26}$$

In order for the series to properly express the induced flow, w, the Ψ_j^r must be a complete set of functions which are linearly independent. The test functions are chosen as

$$\Psi_j^r = \phi_j^r(\bar{r}) = \frac{1}{\nu} \bar{P}_j^r(\nu) \tag{27}$$

where $\nu = \sqrt{1-\bar{r}}$ and $\bar{P}_n^m(\nu) = (-1)^m \bar{P}_n^m / \rho_n^m$. The induced flow then becomes

$$w(\bar{r}, \psi, \bar{t}) = \sum_{r=0}^{\infty} \sum_{j=r+1,r+3}^{\infty} \phi_j^r \left[\alpha_j^r(\bar{t}) \cos r\psi + \beta_j^r(\bar{t}) \sin r\psi \right] \tag{28}$$

The expansion functions $\phi_j^r(\bar{r})$ are determined by

$$\phi_j^r(\bar{r}) = \sqrt{(2j+1)H_j^r} \sum_{q=r,r+2}^{j-1} \bar{r}^q \frac{(-1)^{(q-r)/2}(j+q)!!}{(q-r)!!(q+r)!!(j-q-1)!!}, \tag{29}$$

where

$$H_j^r = \frac{(j+r-1)!!(j-r-1)!!}{(j+r)!!(j-r)!!}. \tag{30}$$

The wake state equations become

$$\frac{2}{\pi} H_n^m \alpha_j^{r+} + L_c^{-1} V_n^m \alpha_j^r = \frac{1}{2} \tau_n^{mc} \tag{31}$$

$$\frac{2}{\pi} H_n^m \beta_j^{r+} + L_s^{-1} V_n^m \beta_j^r = \frac{1}{2} \tau_n^{ms} \tag{32}$$

The velocities matrix V_n^m is diagonal and is constructed using the following

$$V_1^0 = \sqrt{\mu^2 + (\lambda + \nu)^2} \tag{33}$$

$$V_n^m = \frac{\mu^2 + (\lambda + 2\nu)(\lambda + \nu)}{V_1^0} \tag{34}$$

The average induced velocity is the source of nonlinearity and is related to the first inflow state as

$$\nu = \sqrt{3}\ \alpha_1^0 \tag{35}$$

The blade loads form the generalized forces. In non-dimensional form,

$$\tau_n^{0c} = \frac{1}{2\pi}\sum_{q=1}^{N}\left[\int_0^1 \frac{L_q}{\rho\Omega^2 R^3}\phi_n^0(\bar{r})\mathrm{d}\bar{r}\right] \tag{36}$$

$$\tau_n^{mc} = \frac{1}{\pi}\sum_{q=1}^{N}\left[\int_0^1 \frac{L_q}{\rho\Omega^2 R^3}\phi_n^m(\bar{r})\mathrm{d}\bar{r}\right]\cos m\psi_q \tag{37}$$

$$\tau_n^{ms} = \frac{1}{\pi}\sum_{q=1}^{N}\left[\int_0^1 \frac{L_q}{\rho\Omega^2 R^3}\phi_n^m(\bar{r})\mathrm{d}\bar{r}\right]\sin m\psi_q \tag{38}$$

The wake influence coefficients

$$L_{jn}^{0mc} = X^m\Gamma_{jn}^{0m} \tag{39}$$

$$L_{jn}^{rmc} = \left[X^{|m-r|} + (-1)^{\ell}X^{|m+r|}\right]\Gamma_{jn}^{rm} \tag{40}$$

$$L_{jn}^{rms} = \left[X^{|m-r|} - (-1)^{\ell}X^{|m+r|}\right]\Gamma_{jn}^{rm} \tag{41}$$

where $\ell = min(r,m)$ and $X = \tan(\chi/2)$ and

$$\Gamma_{jn}^{rm} = \frac{2(-1)^{(n+j-2r)/2}\sqrt{(2n+1)(2j+1)}}{\sqrt{H_j^r H_n^m}(j+n)(j+n+2)\left[(j-n)^2-1\right]} \tag{42}$$

$$\Gamma_{jn}^{rm} = \frac{\mathrm{sign}(r-m)}{\sqrt{(2n+1)(2j+1)}}, \tag{43}$$

$$\Gamma_{jn}^{rm} = 0. \tag{44}$$

The following table outlines the number of spatial modes corresponding to the number of radial shape function. Previous work has shown that simultaneously increasing the number of shape functions with the modes provides the most suitable number and distribution of wake states.

The wake influence coefficients matrix takes on the following form. Here, the 0^{th}, 1^{st} and 2^{nd} modes are selected each with 3 Legendre functions. This gives a 9×9 matrix of coefficients but distinctly partitioned with 3 rows and columns of sub-matrices with each partition row/column representing a single mode.

$$L_{jn}^{rmc} = \left[\begin{array}{ccc|ccc|ccc}
L_{11}^{00} & L_{13}^{00} & L_{15}^{00} & L_{12}^{01} & L_{14}^{01} & L_{16}^{01} & L_{13}^{02} & L_{15}^{02} & L_{17}^{02} \\
L_{31}^{00} & L_{33}^{00} & L_{35}^{00} & L_{32}^{01} & L_{34}^{01} & L_{36}^{01} & L_{33}^{02} & L_{35}^{02} & L_{37}^{02} \\
L_{51}^{00} & L_{53}^{00} & L_{55}^{00} & L_{52}^{01} & L_{54}^{01} & L_{56}^{01} & L_{53}^{02} & L_{55}^{02} & L_{57}^{02} \\
\hline
L_{21}^{10} & L_{23}^{10} & L_{25}^{10} & L_{22}^{11} & L_{24}^{11} & L_{26}^{11} & L_{23}^{12} & L_{25}^{12} & L_{27}^{12} \\
L_{41}^{10} & L_{43}^{10} & L_{45}^{10} & L_{42}^{11} & L_{44}^{11} & L_{46}^{11} & L_{43}^{12} & L_{45}^{12} & L_{47}^{12} \\
L_{61}^{10} & L_{63}^{10} & L_{65}^{10} & L_{62}^{11} & L_{64}^{11} & L_{66}^{11} & L_{63}^{12} & L_{65}^{12} & L_{67}^{12} \\
\hline
L_{31}^{20} & L_{33}^{20} & L_{35}^{20} & L_{32}^{21} & L_{34}^{21} & L_{36}^{21} & L_{33}^{22} & L_{35}^{22} & L_{37}^{22} \\
L_{51}^{20} & L_{53}^{20} & L_{55}^{20} & L_{52}^{21} & L_{54}^{21} & L_{56}^{21} & L_{53}^{22} & L_{55}^{22} & L_{57}^{22} \\
L_{71}^{20} & L_{73}^{20} & L_{75}^{20} & L_{72}^{21} & L_{74}^{21} & L_{76}^{21} & L_{73}^{22} & L_{75}^{22} & L_{77}^{22}
\end{array}\right]. \tag{45}$$

Highest Power of r	m													Total Inflow states
	0	1	2	3	4	5	6	7	8	9	10	11	12	
0	1													1
1	1	1												3
2	2	1	1											6
3	2	2	1	1										10
4	3	2	2	1	1									15
5	3	3	2	2	1	1								21
6	4	3	3	2	2	1	1							28
7	4	4	3	3	2	2	1	1						36
8	5	4	4	3	3	2	2	1	1					45
9	5	5	4	4	3	3	2	2	1	1				55
10	6	5	5	4	4	3	3	2	2	1	1			66
11	6	6	5	5	4	4	3	3	2	2	1	1		78
12	7	6	6	5	5	4	4	3	3	2	2	1	1	91

Table 1: Total number of states for a specified number of spatial modes and polynomial order.

References

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