The Princeton beam experiment

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The Princeton beam experiment [1, 2] is a study of the large displacement and rotation behavior of a simple cantilevered beam under a gravity tip load. A straight aluminum (T 7075) beam of length $L = 0.508$ m with a rectangular cross-section of thickness $t = 3.175$ mm and height $h = 12.7$ mm was cantilevered at its root and subjected to a static concentrated load P at its tip.

Figure 1 shows an end-view of the test set-up. An inertial frame of reference is selected as $\mathcal{F}^I = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$ and material frame $\mathcal{F}^B = [\mathbf{O}, \mathcal{B} = (\bar{b}_1, \bar{b}_2, \bar{b}_3)]$ is attached at the beam's root section, which is cantilevered into a bearing that allows rotation of the beam about its reference axis by an angle θ, called the "loading angle." The gravity load applied at the beam tip is acting in the opposite direction of unit vector $\bar{\imath}_3$. Variation of the loading angle from 0 to 90 degrees yields a wide range of nonlinear problems where torsion and bending in two directions are coupled.

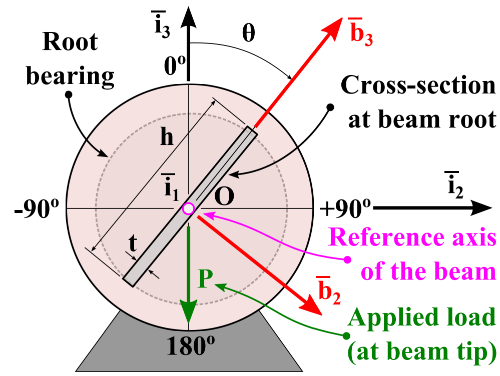

Figure 1: Configuration of the Princeton beam experiment.

Experimental results [1] consist of measurements of the beam's tip deflection along the material unit vectors $\bar{b}_2$ and $\bar{b}_3$, denoted u_2 and u_3, respectively, and called the "flapwise" and "chordwise displacements," respectively. Additionally, the beam's tip twist was also measured. Let $\underline{\underline{R}}^E = \left[\bar{b}_1^E, \bar{b}_2^E, \bar{b}_3^E\right]$ denote the rotation tensor characterizing the rotation of the beam's tip cross-section. In the absence of tip load, $\underline{\underline{R}}^E(P = 0) = \left[\bar{b}_1^E, \bar{b}_2^E, \bar{b}_3^E\right]$, where $\bar{b}_3^{ET} = \left\{0, \sin\theta, \cos\theta\right\}$, and it then follows that $\theta = \arctan(R_{23}^E(P = 0)/R_{33}^E(P = 0))$. Under a tip load P, the orientation of the tip section is defined as $\arctan(R_{23}^E(P)/R_{33}^E(P))$ and the beam's tip twist is defined as

$$\phi = \arctan(\frac{R_{23}^E}{R_{33}^E}) - \theta. \tag{1}$$

The procedure used to measure the twist angle experimentally is detailed in the report by Dowell and Traybar [1].

Data was acquired at loading angles of $\theta = 0, \pm 15, \pm 30, \pm 45, \pm 60, \pm 75, \pm 90$, and 180 degrees. For a perfect system, symmetry implies that the absolute values of the tip displacements and twist should be identical for loading angles $\pm\theta$. In the experimental setting, these measurements differed, providing an estimate of their accuracy. Three loading conditions were used, $P_1 = 4.448$ N (1 lb), $P_2 = 8.896$ N (2 lbs), and $P_3 = 13.345$ N (3 lbs).

1.1 Correlation using linear theory

The linear solution of the problem is found using the shear deformable beam theory described in structural analysis textbooks such as that of Bauchau and Craig [3]. The tip transverse displacement components are

$$u_2^T = \left[\frac{PL^3}{3H_{33}} + \frac{PL}{K_{22}}\right] \sin\theta, \tag{2a}$$

$$u_3^T = \left[\frac{PL^3}{3H_{22}} + \frac{PL}{K_{33}}\right] \cos\theta, \tag{2b}$$

where H_{22} and H_{33} are the bending stiffnesses about material unit vectors $\bar{b}_2$ and $\bar{b}_3$, respectively, and K_{22} and K_{33} the shearing stiffnesses along the same unit vectors, respectively. Of course, for the linear theory, the tip twist vanishes.

For $\theta = 0$ or 180 and $\theta = \pm 90$, the beam undergoes planar deformation and elementary formulæ of Timoshenko beam theory (2) provide the tip deflection in the linear regime. Using the Young's modulus of T 7075 aluminum as $E = 71.7$ GPa and Poisson's ratio $\nu = 0.31$, hand calculations yield $u_2^T = 5.004$ and $u_3^T = 80.034$ mm for the chordwise and flapwise tip displacements, respectively, at loading level P_1. This compares favorably with experimental measurements of $u_2^T = 5.3594$ and $u_3^T = 77.635$ mm, respectively, resulting in -6.6 and +3.1% error, respectively. In this effort, the dimensions of the cross-section were adjusted slightly to achieve good correlation between measurements and predictions of linear theory in these two cases. The following data was used: $L = 0.508$ m, $t = 3.2024$, height $h = 12.377$ mm, $E = 71.7$ GPa, Poisson's ratio $\nu = 0.31$, and shear modulus $G = E/2(1+\nu) = 27.37$ GPa. These physical properties translate to the sectional stiffness properties listed in table 1 and the sectional mass properties are as follows: mass per unit span $m_{00} = 0.1062$ kg/m, moments of inertia per unit span $m_{22} = 1.356$ and $m_{33} = 0.09078$ mg·m^2/m.

Table 1: Sectional stiffness properties of the Princeton beam

	Axial S [MN]	**Shearing** K_{22} [MN]	**Shearing** K_{33} [MN]	**Torsional** H_{11} [N·m^2]	**Bending** H_{22} [N·m^2]	**Bending** H_{33} [N·m^2]
Beam	2.842	0.6401	0.9039	3.103	36.28	2.429

Because the distributed mass of the beam is far smaller than the applied tip weight, it was neglected in the simulations. Consequently, rather than rotating the beam, it was kept at a fixed, vertical orientation at the root, and the direction of the applied load was varied from 0 to 90 degrees.

Experimental results are summarized in figs. 2, 3, and 4 for the tip flapwise, chordwise, and twist for the three loading conditions. For each figure, experimental results are plotted together with the predictions of the geometrically exact beam formulation.

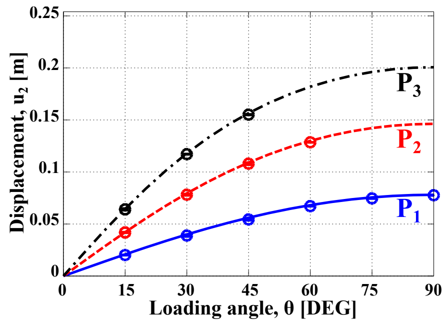

Figure 2: Flapwise displacement at the beam tip versus loading angle for three loading conditions. Experimental measurements: ◯ with error bars. Predictions of geometrically exact beam formulation: solid line, P_1, dashed line, P_2, dashed-dotted line, P_3.

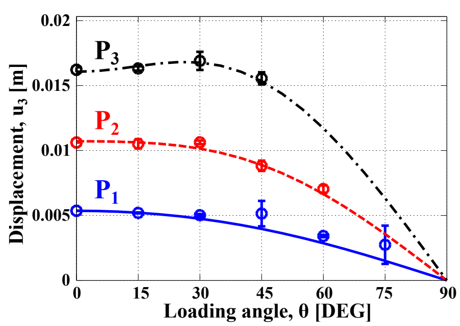

Figure 3: Chordwise displacement at the beam tip versus loading angle for three loading conditions. Experimental measurements: ◯ with error bars. Predictions of geometrically exact beam formulation: solid line, P_1, dashed line, P_2, dashed-dotted line, P_3.

References

[1] E. H. Dowell and J. J. Traybar. An experimental study of the nonlinear stiffness of a rotor blade undergoing flap, lag, and twist deformations. Aerospace and Mechanical Science Report 1257, Princeton University, 1975.

[2] E.H. Dowell, J. Traybar, and D.H. Hodges. An experimental-theoretical correlation study of non-linear bending and torsion deformations of a cantilever beam. *Journal of Sound and Vibration*, 50(4):533–544, February 1977.

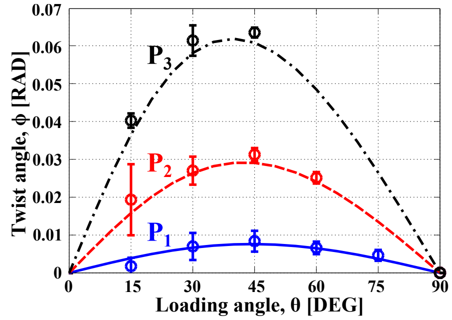

Figure 4: Flapwise displacement at the beam tip versus loading angle for three loading conditions. Experimental measurements: ○ with error bars. Predictions of geometrically exact beam formulation: solid line, P_1, dashed line, P_2, dashed-dotted line, P_3.

[3] O.A. Bauchau and J.I. Craig. *Structural Analysis with Application to Aerospace Structures*. Springer, Dordrecht, Heidelberg, London, New-York, 2009.