Dymore User's Manual
Definition of material properties

Contents

1 Material general properties

The general physical properties of materials are defined in this section. This involves the definition of material density, and the material symmetric type. Three options of material symmetric types are considered here, *isotropic*, *orthotropic*, or *transversely isotropic* materials. The defined material symmetric type decides the definition method for the material stiffness and viscosity properties.

2 Material stiffness properties

The material stiffness properties are defined in this section. Three types of linear elastic materials are considered here, *isotropic*, *orthotropic*, or *transversely isotropic* materials [1, 2, 3].

Linear elastic materials fall into three categories: isotropic, orthotropic, or transversely isotropic. Orthotropic materials possess two orthogonal planes of material property symmetry, implying the existence of a third as illustrated in fig. 1. Transversely isotropic materials feature one plane of material isotropy, *i.e.* properties are identical in all directions in this plane. Typically, advanced composite materials are transversely isotropic. Finally, isotropic material have identical properties in all directions. In each case, a **material basis**, $\mathcal{E}^+ = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$, is defined that reflects the possible existence of various planes of symmetry and/or orthotropy, as illustrated in fig. 1.

The constitutive laws for linear elastic materials will be cast as linear relationships between stress and strain components. The array of *strain components*, resolved in the material basis, is defined as

$$\underline{\epsilon}^{+T} = \lfloor \epsilon_1^+, \epsilon_2^+, \epsilon_3^+, \gamma_{23}^+, \gamma_{13}^+, \gamma_{12}^+ \rfloor, \tag{1}$$

where ϵ_1^+, ϵ_2^+ and ϵ_3^+ are the axial strains components along unit vectors $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively. The corresponding engineering shear strains components are γ_{23}^+, γ_{13}^+ and γ_{12}^+. Notation $(\cdot)^+$ indicates tensor components resolved in basis $\mathcal{E}^+$. The array of *stress components*, resolved in the material basis, is

$$\underline{\sigma}^{+T} = \lfloor \sigma_1^+, \sigma_2^+, \sigma_3^+, \tau_{23}^+, \tau_{13}^+, \tau_{12}^+ \rfloor, \tag{2}$$

where σ_1^+, σ_2^+ and σ_3^+ are the axial stress components along unit vectors $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively; the corresponding shearing stresses are τ_{23}^+, τ_{13}^+ and τ_{12}^+.

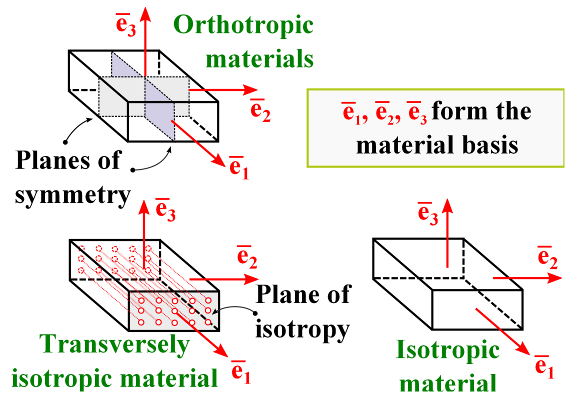

Figure 1: Orthotropic, transversely isotropic and isotropic materials.

A linear elastic material is characterized by the following constitutive laws

$$\underline{\sigma}^+ = \underline{\underline{C}}^+ \underline{\epsilon}^+, \tag{3a}$$

$$\underline{\epsilon}^+ = \underline{\underline{S}}^+ \underline{\sigma}^+, \tag{3b}$$

where $\underline{\underline{C}}^+$ is a 6×6 stiffness matrix and $\underline{\underline{S}}^+$ a 6×6 compliance matrix. Clearly, these two matrices are the inverse of each other, *i.e.*

$$\underline{\underline{C}}^{+-1} = \underline{\underline{S}}^+. \tag{4}$$

The strain energy, A, stored in a differential element of the material is

$$A = \frac{1}{2}\,\underline{\epsilon}^{+T}\underline{\sigma}^+ = \frac{1}{2}\,\underline{\epsilon}^{+T}\underline{\underline{C}}^+\underline{\epsilon}^+ = \frac{1}{2}\,\underline{\sigma}^T\underline{\underline{S}}^+\underline{\sigma}^+. \tag{5}$$

The stored strain energy is a positive quantity for whatever deformation or stress state the material is subjected to. This implies that both stiffness and compliance matrices are symmetric and definite positive.

2.1 Orthotropic materials

An *orthotropic material* has at least two orthogonal planes of material property symmetry. For these materials, the compliance matrix takes the following form

$$\underline{\underline{S}}^+ = \begin{bmatrix} 1/E_1^+ & -\nu_{21}^+/E_2^+ & -\nu_{31}^+/E_3^+ & 0 & 0 & 0 \\ -\nu_{12}^+/E_1^+ & 1/E_2^+ & -\nu_{32}^+/E_3^+ & 0 & 0 & 0 \\ -\nu_{13}^+/E_1^+ & -\nu_{23}^+/E_2^+ & 1/E_3^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12}^+ \end{bmatrix}. \tag{6}$$

The stiffness coefficients appearing in this compliance matrix are three distinct Young's moduli, E_1^+, E_2^+, and E_3^+, three Poisson's ratios, ν_{12}^+, ν_{13}^+, and ν_{23}^+, and three shearing moduli, G_{12}^+, G_{13}^+, and G_{23}^+. Because this matrix is symmetric, the following relationships hold

$$\nu_{23}^+/E_2^+ = \nu_{32}^+/E_3^+, \quad \nu_{31}^+/E_3^+ = \nu_{13}^+/E_1^+, \quad \nu_{21}^+/E_2^+ = \nu_{12}^+/E_1^+. \tag{7}$$

In view of this relationship, the material stiffness properties are characterized by three distinct **Young's moduli**: E_1^+, E_2^+ and E_3^+ along unit vectors $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively; three **Poisson's ratios**: ν_{12}^+, ν_{13}^+ and ν_{23}^+; and three **shearing moduli**: G_{12}^+, G_{13}^+ and G_{23}^+. Thus, for an *orthotropic material*, the following *nine* properties are required: *(1)* **Young's moduli:** E_1^+, E_2^+ and E_3^+; *(2)* **Shear moduli:** G_{12}^+, G_{13}^+ and G_{23}^+; *(3)* **Poisson's ratios:** ν_{12}^+, ν_{13}^+ and ν_{23}^+.

2.2 Transversely isotropic materials

A *transversely isotropic material* is an orthotropic material, additionally presenting a plane of material property isotropy. As illustrated in fig. 1, plane $(\bar{e}_2, \bar{e}_3)$ will be selected as the plane of isotropy. In this case, $E_3^+ = E_2^+$, $G_{13}^+ = G_{12}^+$ and $\nu_{13}^+ = \nu_{12}^+$: in view of the isotropy in the $(\bar{e}_2, \bar{e}_3)$ plane, the subscripts $(.)_2$ and $(.)_3$ can be interchanged. Furthermore, the isotropy of plane $(\bar{e}_2, \bar{e}_3)$ implies $G_{23}^+ = E_2^+/[2(1+\nu_{23}^+)]$. For these materials, the compliance matrix takes the following form

$$\underline{\underline{S}}^+ = \begin{bmatrix} 1/E_1^+ & -\nu_{12}^+/E_1^+ & -\nu_{12}^+/E_1^+ & 0 & 0 & 0 \\ -\nu_{12}^+/E_1^+ & 1/E_2^+ & -\nu_{23}^+/E_2^+ & 0 & 0 & 0 \\ -\nu_{12}^+/E_1^+ & -\nu_{23}^+/E_2^+ & 1/E_2^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu_{23}^+)/E_2^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{12}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12}^+ \end{bmatrix}. \quad (8)$$

For transversely isotropic elastic materials, the following *five* properties are required: *(1)* **Young's moduli:** E_1^+ and E_2^+; *(2)* **Shear moduli:** G_{12}^+, *(3)* **Poisson's ratios:** ν_{12}^+ and ν_{23}^+.

2.3 Isotropic materials

An *isotropic material* is a material that presents identical properties in all directions. In this case, the isotropy of the material implies $E_1^+ = E_2^+ = E_3^+ = E$, $\nu_{12}^+ = \nu_{13}^+ = \nu_{23}^+ = \nu$, and $G_{12}^+ = G_{13}^+ = G_{23}^+ = E/[2(1+\nu)]$. For these materials, the compliance matrix takes the following form

$$\underline{\underline{S}}^+ = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu)/E & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu)/E & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)/E \end{bmatrix}. \quad (9)$$

For isotropic materials, the following *two* properties are required: *(1)* **Young's modulus:** E, *(2)* **Poisson's ratios:** ν.

3 Material viscoelasticity properties

3.1 Generalized Maxwell model

The physical viscoelastic properties of materials are defined in this section. These properties are based on the Generalized Maxwell model [?].

3.1.1 One-dimensional generalized Maxwell model

One-dimensional rheological models [4, 5] are often used to introduce the concepts associated with viscoelasticity. Typically, these models involve serial or parallel combinations of linear or nonlinear springs and dashpots to form increasingly complex models, such as the Kelvin, Maxwell, or Zener models, among many others. For instance, fig. 2 depicts the generalized Maxwell model. The spring elements characterize the elastic behavior of the material, whereas its energy dissipation characteristics are described by the dashpot elements.

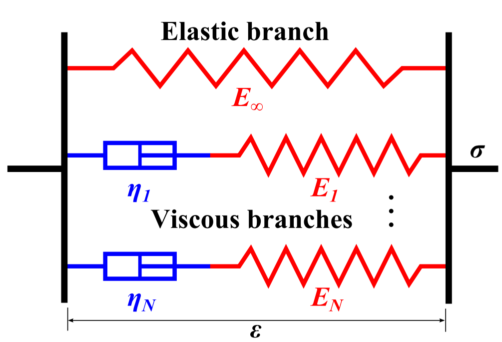

Figure 2: Schematic of the generalized Maxwell model.

The generalized Maxwell model depicted schematically in fig 2 consists of an elastic spring (the "elastic branch") in parallel with one or more Maxwell fluid elements (the "viscous branches") and generalizes the Zener model, also known as the standard linear solid model. Assuming the device to be of unit area and length, forces and elongations can be identified with stresses and strains, respectively. The spring stiffness constants are denoted $E_\infty > 0$ and $E_b > 0$, $b = 1, \ldots, N_b$, where N_b is the number of Maxwell fluid elements. The dashpot constants are denoted $\eta_b > 0$, $b = 1, \ldots, N_b$.

Elementary mechanics yields the constitutive equations for the various components of the system. For the elastic branch, $s_\infty = E_\infty \epsilon$, where s_∞ is the stress in the elastic branch and ϵ the device strain. The stress in a typical viscous branch is $s_b = \eta_b \dot{\alpha}_b = E_b(\epsilon - \alpha_b)$, where the first equation provides the constitutive equation for the dashpot, the second that for the elastic spring, and α_b can be interpreted as the strain in the dashpot. Notation $(\cdot)^{\cdot}$ indicates a derivative with respect to time. Finally, the total stress, σ, is found by summing up the stresses in the branches, $\sigma = s_\infty + \sum_{b=1}^{N_b} s_b$. Introducing the constitutive relationships yields

$$\sigma = E_0 \epsilon - \sum_{b=1}^{N_b} \sigma_b, \tag{10}$$

where $\sigma_b = E_b \alpha_b$ are internal stress states of the model and the initial modulus, E_0, is defined as

$$E_0 = E_\infty + \sum_{b=1}^{N_b} E_b. \tag{11}$$

Because the stresses in the spring and dashpot of each viscous branch are identical, $\eta_b \dot{\alpha}_b = E_b(\epsilon - \alpha_b)$, and hence, the internal states must satisfy the following evolution equation

$$\dot{\sigma}_b + \frac{1}{\tau_b}\sigma_b = \frac{E_b}{\tau_b}\epsilon, \quad b = 1, 2, \ldots, N_b, \tag{12}$$

where the relaxation times, τ_b, are defined as

$$\tau_b = \frac{\eta_b}{E_b}. \tag{13}$$

The first-order differential evolution equation (12) must satisfy the following initial condition, $\lim_{t \to -\infty} \sigma_b = 0$.

Because the evolution equation (12) can be recast as $\mathrm{d}[\exp(t/\tau_b)\sigma_b]/\mathrm{d}t = (E_b/\tau_b)\exp(t/\tau_b)\epsilon$, integration over time yields $\sigma_b = (E_b/\tau_b)\int_{-\infty}^{t} \exp[-(t-s)/\tau_b]\epsilon(s)\ \mathrm{d}s$. Assuming that $\epsilon(t) \to 0$ as $t \to -\infty$, integration by parts then leads to

$$\sigma_b(t) = E_b \epsilon(t) - E_b \int_{-\infty}^{t} e^{-(t-s)/\tau_b} \dot{\epsilon}(s)\ \mathrm{d}s. \tag{14}$$

The stress convolution integral then follows from introducing eq. (14) into eq. (10) to find

$$\sigma(t) = \int_{-\infty}^{t} G(t-s)\dot{\epsilon}(s)\ \mathrm{d}s, \tag{15}$$

where the relaxation function, $G(t)$, is defined as

$$G(t) = E_\infty + \sum_{b=1}^{N_b} E_b e^{-t/\tau_b}. \tag{16}$$

In summary, the one-dimensional generalized Maxwell model can be expressed by convolution integral (15), where the relaxation function is defined by eq. (16). Alternatively, the same model can be represented by eq. (10), where the internal variables satisfy the differential equations of evolution (12). These two formulations of the same model will be used in the sequel.

3.1.2 Three-dimensional generalized Maxwell model

Following the suggested approach of Simo and Hughes [6], the one-dimensional constitutive law for viscoelastic material, eq. (15), can be extended to three-dimensional form by writing the following convolution integral,

$$\underline{\sigma}^+(t) = \int_{-\infty}^{t} \underline{\underline{G}}^+(t-s)\dot{\underline{\gamma}}^+(s)\ \mathrm{d}s, \tag{17}$$

where arrays $\underline{\sigma}^+$ and $\underline{\gamma}^+$ store the six components of the convected Cauchy stress and Green-Lagrange strain tensors, respectively, both resolved in the material basis and the relaxation matrix function, $\underline{\underline{G}}^+(t)$, echoes its corresponding scalar equivalent defined by eq. (16),

$$\underline{\underline{G}}^+(t) = \underline{\underline{\mathcal{C}}}^+_\infty + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}^+_b e^{-t/\tau_b}. \tag{18}$$

Relaxation times, τ_b, $b = 1, 2, \ldots, N_b$, and matrices $\underline{\underline{\mathcal{C}}}^+_\infty$ and $\underline{\underline{\mathcal{C}}}^+_b$, $b = 1, 2, \ldots, N_b$, characterize the three-dimensional viscoelastic behavior of the material. Stiffness matrix $\underline{\underline{\mathcal{C}}}^+_\infty$ characterises the elastic behavior of the material and echoes the elastic constitutive law (3a). Note that the viscoelastic material behavior described by eq. (17) reduces to the elastic behavior described by eq. (3a) if $N_b = 0$ or equivalently, for vanishing relaxation times, $\tau_b \to 0$.

Elementary mathematical arguments show that the convolution integral formulation shown in the previous paragraph can be recast in terms of ordinary differential equations. The total stresses become

$$\underline{\sigma}^+(t) = \underline{\underline{\mathcal{C}}}_0^+ \underline{\gamma}^+ - \sum_{b=1}^{N_b} \underline{\sigma}_b^+(t), \tag{19}$$

where the internal variables of the model, $\underline{\sigma}_b^+$, satisfy the following matrix evolution equations

$$\underline{\dot{\sigma}}_b^+ + \frac{1}{\tau_b}\underline{\sigma}_b^+ = \frac{1}{\tau_b}\underline{\underline{\mathcal{C}}}_b^+ \underline{\gamma}^+, \quad b = 1, 2, \ldots, N_b, \tag{20}$$

with the following initial condition $\lim_{t\to-\infty} \underline{\sigma}_b^+(t) = \underline{0}$. Note the direct parallel between equations (19) and (20), written for three-dimensional problems and their counterparts, equations (10) and (12), respectively, written for one-dimensional problems. As expected from eq. (11), the initial stiffness matrix is $\underline{\underline{\mathcal{C}}}_0^+ = \underline{\underline{\mathcal{C}}}_\infty^+ + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+$.

3.1.3 Time domain description

Equation (17) describes the behavior of viscoelastic materials in the time domain when subjected to a strain history, $\underline{\gamma}^+(t)$. The response of the material is characterized by the relaxation function, $\underline{\underline{G}}^+(t)$, which can be interpreted as the time-dependent stiffness matrix of the material. In the generalized Maxwell model, the relaxation function is expanded in Prony series, see eq. (18).

The physical interpretation of the relaxation function is obtained easily. Consider a step input in strain history, $\underline{\gamma}^+(t) = 0$ for $t < 0$, and $\underline{\gamma}^+(t) = \underline{\gamma}^+$ for $t \geq 0$. Introducing this strain history in eq. (17) yields

$$\underline{\sigma}^+(t) = \int_{-\infty}^{t} \underline{\underline{G}}^+(t-s)\underline{\dot{\gamma}}^+(s) \; \mathrm{d}s = \underline{\underline{G}}^+(t)\underline{\gamma}^+. \tag{21}$$

Clearly, the relaxation function describes the stress history after the application of the step input in strain. The instantaneous stiffness matrix at time $t = 0$ is $\underline{\underline{\mathcal{C}}}_0^+ = \underline{\underline{\mathcal{C}}}_\infty^+ + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+$, whereas the long-term steady stiffness matrix is $\underline{\underline{\mathcal{C}}}_\infty^+$ when $t \to \infty$.

In practice, material viscoelastic properties can be obtained from a stress relaxation experiment: a strain step input is applied to the material and the measurement of the resulting stress provides the stress relaxation function. Curve fitting techniques are then used to extract Prony series from the experimental data.

3.1.4 Relaxation function in frequency domain

In contrast to the time domain characterization described in the previous section, viscoelastic material can also be described in the frequency domain. Consider the following harmonic strain input,

$$\underline{\gamma}^+(t) = \underline{\gamma}_0^+ e^{i\omega t}, \tag{22}$$

where ω is the test frequency and $i = \sqrt{-1}$. Introducing eqs. (22) and (18) into eq. (17) yields

$$\begin{aligned} \underline{\sigma}^+(t) &= \int_{-\infty}^{t} (\underline{\underline{\mathcal{C}}}_\infty^+ + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+ e^{s-t/\tau_b})(i\omega \underline{\gamma}_0^+ e^{i\omega s}) \; \mathrm{d}s \\ &= \underline{\underline{\mathcal{C}}}_\infty^+ \underline{\gamma}^+(t) + (i\omega e^{i\omega t}\underline{\gamma}_0^+) \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+ \int_0^{+\infty} e^{-s/\tau_b} e^{i\omega s} \; \mathrm{d}s \\ &= \left[\underline{\underline{\mathcal{C}}}_\infty^+ + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+ \frac{i\omega\tau_b}{1+i\omega\tau_b} \right] \underline{\gamma}^+(t). \end{aligned} \tag{23}$$

It is customary to characterize the harmonic response of the material in terms of the storage and loss stiffness matrices, denoted $\underline{\underline{G}}_s(\omega)$ and $\underline{\underline{G}}_\ell(\omega)$, respectively,

$$\underline{\sigma}^+(t) = \left[\underline{\underline{G}}_s(\omega) + i\underline{\underline{G}}_\ell(\omega) \right] \underline{\gamma}^+(t), \tag{24}$$

which characterize the in-phase and out-of-phase stress response. Identifying eqs. (23) and (24) then leads to

$$\begin{aligned} \underline{\underline{G}}_s(\omega) &= \underline{\underline{\mathcal{C}}}_\infty + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+ \frac{(\omega\tau_b)^2}{1+(\omega\tau_b)^2} \\ \underline{\underline{G}}_\ell(\omega) &= \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+ \frac{(\omega\tau_b)}{1+(\omega\tau_b)^2}. \end{aligned} \tag{25}$$

Experimentally, the storage and loss matrices are obtained by dynamic mechanical analysis (DMA). Curve fitting techniques are then used to approximate the experimental data by eq. (25).

3.2 Branch definitions

As depicted in fig. 2, the general Maxwell model contains an elastic branch and N_b viscous branches. The elastic branch properties are defined based on the section of **materials stiffness properties**, while the viscous branches properties are defined according to the **branch definitions** of the material viscoelasticity properties.

Each viscous branch has two characteristic properties, a relaxation time τ_b and a characteristic stiffness matrix $\underline{\underline{\mathcal{C}}}_b$. Matrices $\underline{\underline{\mathcal{C}}}_b$ are defined in different form according to the material symmetric types. Three types of linear viscoelastic materials are considered here, *isotropic*, *orthotropic* and *transversely isotropic* materials.

3.2.1 Orthotropic viscoelastic materials

Not implemented at this time.

3.2.2 Transversely isotropic viscoelastic materials

Not implemented at this time.

3.2.3 Isotropic viscoelastic materials

An *isotropic material* is a material that presents identical properties in all directions. For these materials, two independent parameters are needed to decide the stiffness matrix [4]. Because the shear behavior and bulk behavior are usually considered in viscoelasticity, the stiffness matrix defined here takes the following form,

$$\underline{\underline{\mathcal{C}}}_b^+ = \begin{bmatrix} (3\xi_b+4\eta_b)/3 & (3\xi_b-2\eta_b)/3 & (3\xi_b-2\eta_b)/3 & 0 & 0 & 0 \\ (3\xi_b-2\eta_b)/3 & (3\xi_b+4\eta_b)/3 & (3\xi_b-2\eta_b)/3 & 0 & 0 & 0 \\ (3\xi_b-2\eta_b)/3 & (3\xi_b-2\eta_b)/3 & (3\xi_b+4\eta_b)/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_b & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_b & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_b \end{bmatrix}. \tag{26}$$

Hence, for isotropic materials, the following *three* properties are required for each viscous branch: *(1)* **Relaxation time:** τ_b, *(2)* **Bulk modulus:** ξ_b, *(3)* **Shear modulus:** η_b.

These three required parameters can be obtained by material experiments. The **time domain** and the **frequency domain** experiments are introduced here.

In **time domain** if a relaxation experiment in shear behavior is performed, according to equations (18) and (21), the time-dependent shear modulus becomes,

$$\eta(t) = \eta_\infty + \sum_{b=1}^{N_b} \eta_b e^{-t/\tau_b}, \tag{27}$$

where η_∞ is the equivalent shear modulus defined in eq. (9). Time-dependent shear modulus can be obtained by performing the relaxation experiment. The time-dependent shear modulus can then be approximated by a number of Prony series according to eq. (27), which can be fitted with the experiment data via different data fitting methods, such as least squares. Each Prony term is defined in one viscous branch. In each branch the relaxation time for shear and bulk behavior is assumed to be the same in Dymore. The method to decide the bulk modulus in time domain is similar to that of shearing.

In **frequency domain** if a dynamic mechanical analysis in shear behavior is performed, according to eq. (23), the shear response becomes,

$$\sigma(t) = [\eta_s(\omega) + i\eta_l(\omega)]\,\gamma(t), \tag{28}$$

where $\gamma(t)$ is the harmonic shear strain, $\sigma(t)$ is the shear stress response, $\eta_s(\omega)$ is the storage shear modulus and $\eta_l(\omega)$ is the loss shear modulus.

The absolute magnitude of the shear response is

$$|\sigma| = \sqrt{\eta_s^2(\omega) + \eta_l^2(\omega)}|\gamma|, \tag{29}$$

and the phase lag of the shear response is

$$\phi = \arctan \frac{\eta_l(\omega)}{\eta_s(\omega)}. \tag{30}$$

Measurements of $|\tau|$ and ϕ obtained in the experiment can then be used to define η_s and η_l [7]. Then the storage shear modulus and loss shear modulus can be expressed parallel to eq. (25),

$$\begin{aligned}\eta_s(\omega) &= \eta_\infty + \sum_{b=1}^{N_b} \eta_b^+ \frac{\omega^2\tau_b^2}{1+\omega^2\tau_b^2}\\ \eta_l(\omega) &= \sum_{b=1}^{N_b} \eta_b^+ \frac{\omega\tau_b}{1+\omega^2\tau_b^2}.\end{aligned} \tag{31}$$

In each branch the relaxation time for shear and bulk response is assumed to be the same in Dymore. The method for bulk behavior is also analogous to that of the shear behavior.

In general the three required parameters can be obtained from relaxation experiment according to eq. (27), or from harmonic dynamic mechanical analysis according to eq. (31).

4 Failure criterion type

4.1 Hoffmann's criterion

This criterion is used for *transversely isotropic material only*. At failure, the following equation is satisfied

$$s_1^2 - F_{12}s_1s_2 + s_2^2 + s_6^2 + F_1s_1 + F_2s_2 = 1, \tag{32}$$

where $s_1 = \sigma_1/(\sigma_1^{\rm aT}\sigma_1^{\rm aC})^{1/2}$, $s_2 = \sigma_2/(\sigma_2^{\rm aT}\sigma_2^{\rm aC})^{1/2}$ and $s_6 = \tau_{12}/\tau_{12}^{\rm a}$, σ_1 and σ_2 are the **stresses along the material axes** $\bar{e}_1$ and $\bar{e}_2$, respectively, and τ_{12} the corresponding shear stress. The allowable tensile and compressive stresses along unit vector $\bar{e}_1$ are denoted $\sigma_1^{\rm aT}$ and $\sigma_1^{\rm aC}$, respectively. Similarly, the allowable tensile and compressive stresses along unit vector $\bar{e}_2$ are denoted $\sigma_2^{\rm aT}$ and $\sigma_2^{\rm aC}$, respectively. Finally, the allowable shear stresses in plane $(\bar{e}_1, \bar{e}_2)$ is denoted $\tau_{12}^{\rm a}$. All these quantities are defined in section 5. The two coefficients F_1 and F_2 are

$$F_1 = (\sigma_1^{\rm aC} - \sigma_1^{\rm aT})/(\sigma_1^{\rm aT}\sigma_1^{\rm aC})^{1/2}, \tag{33a}$$

$$F_2 = (\sigma_2^{\rm aC} - \sigma_2^{\rm aT})/(\sigma_2^{\rm aT}\sigma_2^{\rm aC})^{1/2}, \tag{33b}$$

$$F_{12} = (\sigma_2^{\rm aT}\sigma_2^{\rm aC})^{1/2}/(\sigma_1^{\rm aT}\sigma_1^{\rm aC})^{1/2}. \tag{33c}$$

4.2 Maximum strain criterion

This criterion is used for *transversely isotropic material only*. At failure, one of the following equations is satisfied.

$$\epsilon_1 = \begin{cases}\sigma_1^{\rm aT}/E_1^+ & \text{if} \quad \epsilon_1 > 0,\\ \sigma_1^{\rm aC}/E_1^+ & \text{if} \quad \epsilon_1 < 0,\end{cases} \tag{34a}$$

$$\epsilon_2 = \begin{cases}\sigma_2^{\rm aT}/E_2^+ & \text{if} \quad \epsilon_2 > 0,\\ \sigma_2^{\rm aC}/E_2^+ & \text{if} \quad \epsilon_2 < 0,\end{cases}, \tag{34b}$$

$$\gamma_{12} = \tau_{12}^{\rm a}/G_{12}^+, \tag{34c}$$

where ϵ_1 and ϵ_2 are the **strains along the material axes** $\bar{e}_1$ and $\bar{e}_2$, respectively, and γ_{12} the corresponding shear strain. The allowable tensile and compressive stresses along unit vector $\bar{e}_1$ are denoted $\sigma_1^{\rm aT}$ and $\sigma_1^{\rm aC}$, respectively. Similarly, the allowable tensile and compressive stresses along unit vector $\bar{e}_2$ are denoted $\sigma_2^{\rm aT}$ and $\sigma_2^{\rm aC}$, respectively. Finally, the allowable shear stresses in plane $(\bar{e}_1, \bar{e}_2)$ is denoted $\tau_{12}^{\rm a}$. All these quantities are defined in section 5.

4.3 Maximum stress criterion

This criterion is used for *transversely isotropic material only*. At failure, one of the following equations is satisfied.

$$\sigma_1 = \begin{cases}\sigma_1^{\rm aT} & \text{if} \quad \sigma_1 > 0,\\ \sigma_1^{\rm aC} & \text{if} \quad \sigma_1 < 0,\end{cases} \tag{35a}$$

$$\sigma_2 = \begin{cases}\sigma_2^{\rm aT} & \text{if} \quad \sigma_2 > 0,\\ \sigma_2^{\rm aC} & \text{if} \quad \sigma_2 < 0,\end{cases} \tag{35b}$$

$$\tau_{12} = \tau_{12}^{\rm a}, \tag{35c}$$

where σ_1 and σ_2 are the **stresses along the material axes** $\bar{e}_1$ and $\bar{e}_2$, respectively, and τ_{12} the corresponding shear stress. The allowable tensile and compressive stresses along unit vector $\bar{e}_1$ are denoted $\sigma_1^{\rm aT}$ and $\sigma_1^{\rm aC}$, respectively. Similarly, the allowable tensile and compressive stresses along unit vector $\bar{e}_2$ are denoted $\sigma_2^{\rm aT}$ and $\sigma_2^{\rm aC}$, respectively. Finally, the allowable shear stresses in plane $(\bar{e}_1, \bar{e}_2)$ is denoted $\tau_{12}^{\rm a}$. All these quantities are defined in section 5.

4.4 Tsai-Wu's criterion

This criterion is used for *transversely isotropic material only.* At failure, the following equation is satisfied

$$s_1^2 - s_1 s_2 + s_2^2 + s_6^2 + F_1 s_1 + F_2 s_2 = 1,$$

where $s_1 = \sigma_1/(\sigma_1^{aT}\sigma_1^{aC})^{1/2}$, $s_2 = \sigma_2/(\sigma_2^{aT}\sigma_2^{aC})^{1/2}$ and $s_6 = \tau_{12}/\tau_{12}^{a}$, σ_1 and σ_2 are the **stresses along the material axes** $\bar{e}_1$ and $\bar{e}_2$, respectively, and τ_{12} the corresponding shear stress. The allowable tensile and compressive stresses along unit vector $\bar{e}_1$ are denoted σ_1^{aT} and σ_1^{aC}, respectively. Similarly, the allowable tensile and compressive stresses along unit vector $\bar{e}_2$ are denoted σ_2^{aT} and σ_2^{aC}, respectively. Finally, the allowable shear stresses in plane $(\bar{e}_1, \bar{e}_2)$ is denoted τ_{12}^{a}. All these quantities are defined in section 5. The two coefficients F_1 and F_2 are

$$F_1 = (\sigma_1^{aC} - \sigma_1^{aT})/(\sigma_1^{aT}\sigma_1^{aC})^{1/2}, \quad (36a)$$

$$F_2 = (\sigma_2^{aC} - \sigma_2^{aT})/(\sigma_2^{aT}\sigma_2^{aC})^{1/2}. \quad (36b)$$

4.5 Von Mises' criterion

This criterion is used for *isotropic materials only.* At failure, the following equation is satisfied

$$\frac{1}{2}\left[(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2\right] = 1,$$

where $s_1 = \sigma_1/\sigma^{aT}$, $s_2 = \sigma_2/\sigma^{aT}$ and $s_3 = \sigma_3/\sigma^{aT}$, σ_1, σ_2 and σ_3 are **the principal stresses**, and σ^{aT} the allowable stress in tension defined in section 5. Note that since the material is assumed to be isotropic, its strength is identical in all directions; furthermore, its compressive and tensile strengths are assumed to be identical.

5 Allowable stresses

In general, the strength properties of a material can be defined by nine different stress values. *(1) Allowable Tensile Stress*: σ_1^{aT}, σ_2^{aT} and σ_3^{aT}, along the material axes $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively.*(2) Allowable Compressive Stress*: σ_1^{aC}, σ_2^{aC} and σ_3^{aC}, along the material axes $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively. *(3)Allowable Shear Stress*: τ_{12}^{a}, τ_{13}^{a} and τ_{23}^{a}.

For different types of materials, the required number of strength properties are different.

1. For an *orthotropic material*, all *nine* strength properties are required: *(1)* **Tension strength:** σ_1^{aT}, σ_2^{aT} and σ_3^{aT}; *(2)* **Compression strength:** σ_1^{aC},σ_2^{aC} and σ_3^{aC}; *(3)* **Shear strength:** τ_{12}^{a}, τ_{13}^{a} and τ_{23}^{a}.

2. For a *transversely isotropic material*, the following *five* properties are required: *(1)* **Tension strength:** σ_1^{aT} and σ_2^{aT}; *(2)* **Compression strength:** σ_1^{aC} and σ_2^{aC}; *(3)* **Shear strength:** τ_{12}^{a}. In this case, $\sigma_3^{aT} = \sigma_2^{aT}$, $\sigma_3^{aC} = \sigma_2^{aC}$ and $\tau_{13}^{a} = \tau_{12}^{a}$: in view of the isotropy in the $(\bar{e}_2, \bar{e}_3)$ plane, the subscripts $(.)_2$ and $(.)_3$ can be interchanged. Furthermore, the isotropy of plane $(\bar{e}_2, \bar{e}_3)$ implies $\tau_{23}^{a} = \sigma_2^{aT}/\sqrt{3}$.

3. For an *isotropic material*, a *single* property is required: **Tension strength:** σ^{a}. In this case, $\sigma_1^{aT} = \sigma_2^{aT} = \sigma_3^{aT} = \sigma^{a}$, $\sigma_1^{aC} = \sigma_2^{aC} = \sigma_3^{aC} = \sigma^{a}$ and $\tau_{12}^{a} = \tau_{13}^{a} = \tau_{23}^{a} = \sigma^{a}/\sqrt{3}$.

6 Rotation of material properties

In section 2, material stiffness properties were defined in the material basis, $\mathcal{E}^+ = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$. When using these properties to evaluate the stiffness matrix of solid elements, it be be necessary to rotate these properties to the reference basis, $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$. The definition of the relative orientation of the material basis with respect to the reference basis is performed based on *layer orientation angles* and two options, the "local basis option" and the "global basis option," are possible to define this relative orientation. These two options are described in sections 6.1 and 6.2, respectively.

6.1 The local basis option

When using the local basis option, three bases are involved in the determination of the orientation of the material basis: the *global reference basis*, $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$, the *solid local basis*, $\mathcal{U} = (\bar{\imath}_1, \bar{u}, \bar{v})$, and the *material basis*, $\mathcal{E}^+ = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$. The orientation of the solid local basis, $\mathcal{U}$, and two orientation angles, β and γ, are used to define the orientation of the material basis, $\mathcal{E}^+$.

A sequence of *three planar rotations* brings the reference basis, $\mathcal{I}$, to the material basis, $\mathcal{E}^+$, as illustrated in fig. 3.

1. The *first planar rotation* is of magnitude α about unit vector $\bar{\imath}_1$ and brings the reference basis, $\mathcal{I}$, to the solid local basis, $\mathcal{U}$. Angle α is determined by the geometry of the finite element.
2. The *second planar rotation* is of magnitude β about unit vector $\bar{\imath}_1$ and brings the solid local basis, $\mathcal{U}$, to basis $\mathcal{B} = (\bar{\imath}_1, \bar{b}_2, \bar{b}_3)$. Angle β is a user input. Because these first two planar rotations take place about the same unit vector, $\bar{\imath}_1$, they can be combined into a single planar rotation of magnitude $(\alpha + \beta)$ about unit vector $\bar{\imath}_1$.
3. The *third planar rotation* is of magnitude γ about unit vector $\bar{b}_3$ and brings basis $\mathcal{B}$, to the material basis, $\mathcal{E}^+$. Angle γ is a user input.

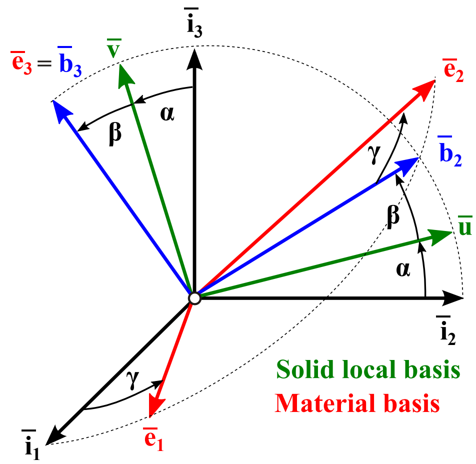

Figure 3: Orientation of the material basis using the **Local Axes** option.

With the help of fig. 4, it is verified readily that the rotation tensor, $\underline{\underline{R}}$ that brings basis $\mathcal{I}$ to basis $\mathcal{E}^+$ is

$$\underline{\underline{R}} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma\cos(\alpha+\beta) & \cos\gamma\cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin\gamma\sin(\alpha+\beta) & \cos\gamma\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \tag{37}$$

Note that positive angles β and γ correspond to positive rotations about axes $\bar{\imath}_1$ and $\bar{b}_3$, respectively, following the right-hand rule. If the layer is a transversely isotropic material such as a unidirectional layer of composite, angle $\beta = 0$ and angle γ corresponds to the fiber orientation angle.

6.2 The global basis option

When using the global basis option, two bases are involved in the determination of the orientation of the material basis: the *global reference basis*, $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ and the *material basis*, $\mathcal{E}^+ = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$. The two orientation angles, β and γ, are used to define the orientation of the material basis, $\mathcal{E}^+$.

A sequence of *two planar rotations* brings the reference basis, $\mathcal{I}$, to the material basis, $\mathcal{E}^+$, as illustrated in fig. 4.

1. The *first planar rotation* is of magnitude β about unit vector $\bar{\imath}_1$ and brings the global reference basis, $\mathcal{I}$, to frame $\mathcal{B} = (\bar{\imath}_1, \bar{b}_2, \bar{b}_3)$. Angle β is a user input.
2. The *second planar rotation* is of magnitude γ about unit vector $\bar{b}_3$ and brings basis $\mathcal{B}$, to the material basis, $\mathcal{E}^+$.

With the help of fig. 4, it is verified readily that the rotation tensor, $\underline{\underline{R}}$ that brings basis $\mathcal{I}$ to basis $\mathcal{E}^+$ is

$$\underline{\underline{R}} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma\cos\beta & \cos\gamma\cos\beta & -\sin\beta \\ \sin\gamma\sin\beta & \cos\gamma\sin\beta & \cos\beta \end{bmatrix} \tag{38}$$

Note that positive angles β and γ correspond to positive rotations about axes $\bar{\imath}_1$ and $\bar{b}_3$, respectively, following the right-hand rule. It is important to note that in this scheme, while the layer orientation depends on the solid local basis, the determination of the material basis orientation is independent of that of the solid local basis.

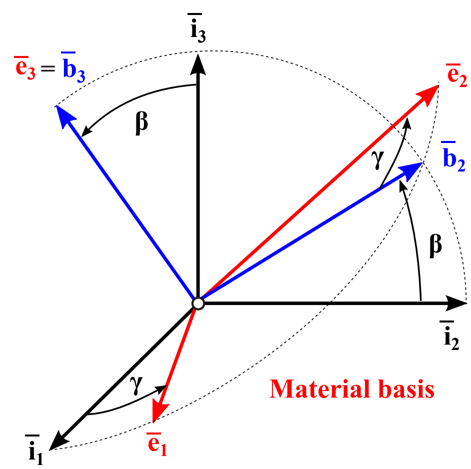

Figure 4: Orientation of the material basis using the "global axes option."

6.3 Rotating material properties

Let the components of the stress tensor resolved in bases $\mathcal{I}$ and $\mathcal{E}$ be denoted $\underline{\underline{\sigma}}$ and $\underline{\underline{\sigma}}^+$, respectively. The following relationship then holds, $\underline{\underline{\sigma}}^+ = \underline{\underline{R}}^T \underline{\underline{\sigma}}\, \underline{\underline{R}}$, where $\underline{\underline{R}}$ are the components of the rotation tensor that brings basis $\mathcal{I}$ to basis $\mathcal{E}$, resolved in basis $\mathcal{I}$ [8]. For the problem at hand, the inverse relationship is desired, $\underline{\underline{\sigma}} = \underline{\underline{R}}\, \underline{\underline{\sigma}}^+ \underline{\underline{R}}^T$.

To simplify the notation, let

$$\underline{\underline{R}} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}, \tag{39}$$

where ℓ_1, ℓ_2, and ℓ_3, m_1, m_2, and m_3, and n_1, n_2, and n_3 are the direction cosines of unit vectors $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3$, respectively. The stress transformation, written in term of the stress array, see eq. (2), then becomes $\underline{\sigma} = \underline{\underline{R}}_\sigma \underline{\sigma}^+$, where

$$\underline{\underline{R}}_\sigma = \begin{bmatrix} \ell_1^2 & m_1^2 & n_1^2 & 2m_1n_1 & 2\ell_1n_1 & 2\ell_1m_1 \\ \ell_2^2 & m_2^2 & n_2^2 & 2m_2n_2 & 2\ell_2n_2 & 2\ell_2m_2 \\ \ell_3^2 & m_3^2 & n_3^2 & 2m_3n_3 & 2\ell_3n_3 & 2\ell_3m_3 \\ \ell_2\ell_3 & m_2m_3 & n_2n_3 & m_2n_3+m_3n_2 & \ell_2n_3+\ell_3n_2 & \ell_2m_3+\ell_3m_2 \\ \ell_1\ell_3 & m_1m_3 & n_1n_3 & m_1n_3+m_3n_1 & \ell_1n_3+\ell_3n_1 & \ell_1m_3+\ell_3m_1 \\ \ell_1\ell_2 & m_1m_2 & n_1n_2 & m_1n_2+m_2n_1 & \ell_1n_2+\ell_2n_1 & \ell_1m_2+\ell_2m_1 \end{bmatrix}. \tag{40}$$

The strain transformation, written in term of the strain array, see eq. (1), becomes $\underline{\epsilon} = \underline{\underline{R}}_\epsilon \underline{\epsilon}^+$, where

$$\underline{\underline{R}}_\epsilon = \begin{bmatrix} \ell_1^2 & m_1^2 & n_1^2 & m_1n_1 & \ell_1n_1 & \ell_1m_1 \\ \ell_2^2 & m_2^2 & n_2^2 & m_2n_2 & \ell_2n_2 & \ell_2m_2 \\ \ell_3^2 & m_3^2 & n_3^2 & m_3n_3 & \ell_3n_3 & \ell_3m_3 \\ 2\ell_2\ell_3 & 2m_2m_3 & 2n_2n_3 & m_2n_3+m_3n_2 & \ell_2n_3+\ell_3n_2 & \ell_2m_3+\ell_3m_2 \\ 2\ell_1\ell_3 & 2m_1m_3 & 2n_1n_3 & m_1n_3+m_3n_1 & \ell_1n_3+\ell_3n_1 & \ell_1m_3+\ell_3m_1 \\ 2\ell_1\ell_2 & 2m_1m_2 & 2n_1n_2 & m_1n_2+m_2n_1 & \ell_1n_2+\ell_2n_1 & \ell_1m_2+\ell_2m_1 \end{bmatrix}. \tag{41}$$

The constitutive laws, written as $\underline{\sigma}^+ = \underline{\underline{C}}^+ \underline{\epsilon}^+$ in eq. (3a) can now be transformed: $\underline{\sigma} = \underline{\underline{R}}_\sigma \underline{\sigma}^+ = \underline{\underline{R}}_\sigma \underline{\underline{C}}^+ \underline{\underline{R}}_\epsilon^{-1} \underline{\epsilon}$, which implies that $\underline{\underline{C}} = \underline{\underline{R}}_\sigma \underline{\underline{C}}^+ \underline{\underline{R}}_\epsilon^{-1}$. Because the stiffness matrix is symmetric in all bases, $\underline{\underline{R}}_\epsilon^{-1} = \underline{\underline{R}}_\sigma^T$, and hence,

$$\underline{\underline{C}} = \underline{\underline{R}}_\sigma \underline{\underline{C}}^+ \underline{\underline{R}}_\sigma^T. \tag{42}$$

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