Dymore User's Manual

Formulation and finite element implementation of force element interface

Contents

1 Introduction

Force elements, such as hydraulic devices or elastomeric dampers, interact with the dynamics of the mechanical system they are connected to. For instance, a helicopter lead-lag damper interacts with rotor blade dynamics; this effect is particularly pronounced on the fundamental blade lead-lag mode. This section describes the coupling of force elements with a structural dynamics model, within the framework of multibody system dynamics [1]. The following sections describe the coupling procedure in terms of the applied structural forces, their time discretization, and the time integration scheme for the equations governing the behavior of force elements.

In general, force elements generate forces, denoted F^a, that are functions of the elongation, ϵ, and elongation rate $\dot{\epsilon}$. The functional dependency, $F^a = F^a(\epsilon, \dot{\epsilon})$, depends on the specific force element to be considered. The elongation is evaluated from the configuration of the force element interface depicted in fig. 1, which shows a hydraulic device as the force element. Of course, another device, such as an elastomeric damper could take the place of the hydraulic device.

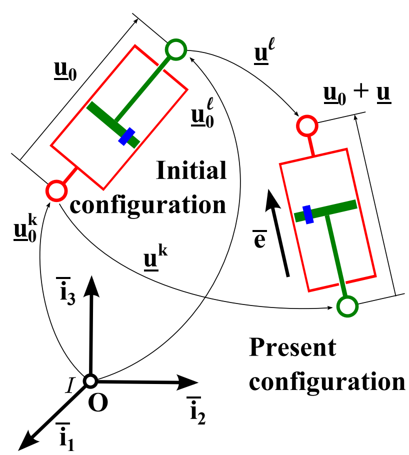

Figure 1: Configuration of the force element interface.

2 Kinematics of the problem

In the initial configuration, the end points of the force element interface are at location $\underline{u}_0^k$ and $\underline{u}_0^\ell$, respectively, with respect to an inertial frame $\mathcal{F} = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$. At those points, the force element interface is connected to a dynamical system of arbitrary topology. In the deformed configuration, the displacements of the end points of the force element interface are $\underline{u}^k$ and $\underline{u}^\ell$, respectively. The relative position of the end points will be denoted $\underline{u}_0 = \underline{u}_0^\ell - \underline{u}_0^k$ and $\underline{u} = \underline{u}^\ell - \underline{u}^k$, in the initial and present configurations, respectively. Note that the rotational degrees of freedom of the structure at the connection points are not involved in this formulation, implying the presence of spherical joints at these points.

The elongation, ϵ, of the device is found to be

$$\epsilon = \|\underline{u}_0 + \underline{u}\| - \|\underline{u}_0\|, \tag{1}$$

and the unit vector along the axis of the force element interface is

$$\bar{e} = \frac{\underline{u}_0 + \underline{u}}{\|\underline{u}_0 + \underline{u}\|}. \tag{2}$$

Finally, the elongation rate is found as

$$\dot{\epsilon} = \bar{e}^T \dot{\underline{u}}. \tag{3}$$

3 Prescribed elongation actuator

The prescribed elongation actuator enforces the rheonomic constraints

$$\mathcal{C} = \epsilon - \hat{h}(t), \tag{4}$$

where $\hat{h}(t)$ is the prescribed elongation of the actuator. Constraint (4) describes the highly idealized situation where the dynamics of the actuator is ignored altogether.

This holonomic constraint is enforced using Lagrange's multiplier method. The potential of the constraint forces is $V^c = \lambda\mathcal{C}$, where λ is Lagrange's multiplier used to enforce this constraint; variation of this potential yields $\delta V^c = \delta\lambda\mathcal{C} + \lambda\delta\mathcal{C}$. The second term represents the virtual work done by the constraint force, $\delta W^c = \lambda\delta\mathcal{C} = \delta\underline{q}^T \underline{F}^c$, where array $\delta\underline{q}$ stores variations of the generalized coordinates associated with this constraint and $\underline{F}^c$ the corresponding constraint forces. Variation of the constraint is expressed as $\delta\mathcal{C} = \underline{B}\delta\underline{q}$, and it follows that $\underline{F}^c = \lambda\underline{B}^T$ where

$$\delta\underline{q} = \begin{Bmatrix} \delta\underline{u}^k \\ \delta\underline{u}^\ell \end{Bmatrix}, \quad \underline{B}^T = \begin{Bmatrix} -\bar{e} \\ \bar{e} \end{Bmatrix}, \quad \underline{F}^c = \lambda\underline{B}^T. \tag{5}$$

Lagrange multiplier λ can be interpreted as the magnitude of the force required to enforce the prescribed elongation constraint.

An increment in the forces of constraint is expressed as $\Delta\underline{F}^c = \underline{\underline{X}}\Delta\underline{q}$, where $\underline{\underline{X}}$ is the *equivalent stiffness matrix* for the constraint. Partial derivatives of the constraint forces yield the following expression for this matrix

$$\underline{\underline{X}} = \lambda \begin{bmatrix} \underline{\underline{D}} & -\underline{\underline{D}} \\ -\underline{\underline{D}} & \underline{\underline{D}} \end{bmatrix}, \tag{6}$$

where matrix $\underline{\underline{D}}$, implicitly defined as $\Delta\bar{e} = \underline{\underline{D}}\,\Delta\underline{u}$, is found easily

$$\underline{\underline{D}} = \frac{1}{\|\underline{u}_0 + \underline{u}\|}\left(\underline{\underline{I}} - \bar{e}\,\bar{e}^T\right), \tag{7}$$

where $\underline{\underline{I}}$ is the identity matrix.

4 The force element interface

If a force element is present, the virtual work it performs is $\delta W = F^a\delta\epsilon$, where F^a is the force generated by the force element. This expression then becomes $\delta W = \delta\underline{q}^T\underline{F}$, where

$$\underline{F} = F^a\underline{B}^T, \tag{8}$$

and $\delta\underline{q}$ and $\underline{B}^T$ are defined by eq. (5). Note the similarity between the two formulations: the Lagrange multiplier, λ, appearing in the prescribed elongation actuator is replaced by the force, F^a, generated by the force element. In both cases, two forces of equal magnitude and opposite sign are applied at the connection points.

Because the expression for the device forces and the governing equations of dynamical systems are nonlinear, the solution process involves iteration and linearization. Linearization of these forces leads to $\Delta \underline{F} = \underline{\underline{K}}\, \Delta \underline{q}$, where the effective stiffness matrix of the device, denoted $\underline{\underline{K}}$, is

$$\underline{\underline{K}} = \begin{bmatrix} \underline{\underline{D}} & -\underline{\underline{D}} \\ -\underline{\underline{D}} & \underline{\underline{D}} \end{bmatrix}, \tag{9}$$

where matrix $\underline{\underline{D}}$ is

$$\underline{\underline{D}} = \frac{F^a}{\|\underline{u}_0 + \underline{u}\|} \underline{\underline{I}} + \left(\frac{\mathrm{d}F^a}{\mathrm{d}\epsilon} - \frac{F^a}{\|\underline{u}_0 + \underline{u}\|} \right) \bar{e}\, \bar{e}^T, \tag{10}$$

This expression requires the evaluation of the derivative of the device force with respect to the elongation, $\mathrm{d}F^a/\mathrm{d}\epsilon$, that could be computed from the governing equations for the device. In general, this process is quite involved. The following approximation was found to be suitable

$$\frac{\mathrm{d}F^a}{\mathrm{d}\epsilon} = \frac{BA_0^2}{V_0} + \frac{BA_1^2}{V_1}, \tag{11}$$

which corresponds to an approximation to the static stiffness of the hydraulic device.

4.1 Time integration of hydraulic equations

The model described in the previous section requires the knowledge of the force acting in the force element. In turn, this requires the solution of the equations governing the behavior of the device. Although the model of the force elements is rather simple, a few first order, nonlinear differential equations, it is a numerically stiff set of equations because of the very high "stiffness" of the hydraulic fluid (for typical systems, the bulk modulus of the fluid is about 1.5 GPa), for instance. Typically, this problem is overcome by using a very small time step for the integration of the force elements. For instance, Welsh [2] used a time step of $\Delta t = 1$ ms to integrate the equations of a helicopter air-oil strut. While this approach is acceptable when dealing with the sole hydraulic equations, it is not practical to integrate both hydraulic and structural dynamics equations with such a small time step because the computational effort would become overwhelming. Consequently, it is imperative to decouple the integration of the two systems: the structural dynamics equations are integrated with a time step dictated by the frequency content of the structural response, whereas the hydraulic equations are integrated with a much smaller time step.

For the force elements, the following strategy is used: the structural dynamics equations are integrated with a time step size denoted Δt. This produces a prediction of the force element elongation. This information is used to integrate the governing equations of the force element using a fourth-order Runge-Kutta integrator [3]. The time step size used in this integrator is $h = \Delta t/N$, *i.e.* N Runge-Kutta steps are performed for each structural time step. The nonlinear solution of the problem is then obtained by iterating between the structural dynamics equations and the force element equations.

5 Inertial effects

Figure 1 shows the configuration of the force element interface. While the forces generated by the force element depend on the nature of the device (hydraulic or elastomeric damper, for instance), the inertia forces caused by the mass of the device can be treated in a generic manner.

As depicted on fig. 2, it is assumed that the mass of the device is concentrated at point **M**, whose position vector, $\underline{r}_M$, is the average of those of points **K** and **L**, $\underline{r}_M = (\underline{r}_K + \underline{r}_L)/2$. The velocity of point **M** then becomes

$$\underline{v}_M = \frac{1}{2}(\dot{\underline{u}}_k + \dot{\underline{u}}_\ell). \tag{12}$$

This definition implicitly assumes that the elongation of the device is small, and hence, point **M** is treated as a material point of a rigid body whose mass equals that of the device. It is further assumed that unit vector $\bar{e}$, defined by eq. (2) and shown in fig. 2, is also a material vector of the same rigid body.

The time derivative of unit vector $\bar{e}$ is found easily as

$$\dot{\bar{e}} = (\underline{\underline{I}} - \bar{e}\,\bar{e}^T)\frac{\dot{\underline{u}}}{\|\underline{u}_0 + \underline{u}\|}. \quad (13)$$

Because unit vector $\bar{e}$ is a material vector of a rigid body, it can be expressed as $\dot{\bar{e}} = \tilde{\omega}\,\bar{e}$, where $\underline{\omega}$ denotes the body's angular velocity vector. The angular velocity vector is the solution of the vector product equation, $\tilde{e}\,\underline{\omega} = -\dot{\bar{e}}$, *i.e.*, $\underline{\omega} = \mu\bar{e} + \tilde{e}\dot{\bar{e}}$, where μ is an arbitrary scalar. Introducing eq. (13) yields the angular velocity vector as $\underline{\omega} = \mu\bar{e} + \tilde{e}\dot{\underline{u}}/\|\underline{u}_0 + \underline{u}\|$, where the first term on the right-hand side implies that the component of the angular velocity vector along unit vector $\bar{e}$ remains undetermined. It is assumed that rotation of the device about its own axis is prevented, leading to the following expression of the angular velocity vector

$$\underline{\omega} = \frac{\tilde{e}\dot{\underline{u}}}{\|\underline{u}_0 + \underline{u}\|}. \quad (14)$$

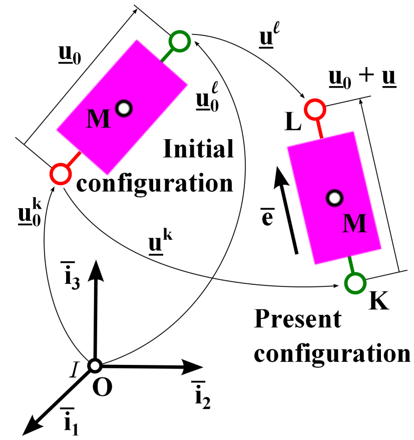

Figure 2: Configuration of the force element interface.

The material basis attached to the device is denoted $\mathcal{B}$ and let $\underline{\underline{R}} = [\bar{e}, \bar{p}, \bar{q}]$ denote the rotation tensor that bring basis $\mathcal{I}$ to basis $\mathcal{B}$. Note that unit vectors $\bar{p}$ and $\bar{q}$ are orthogonal to unit vector $\bar{e}$, but their orientation in the normal plane is not defined. The components of the angular velocity vector in this material basis are then

$$\underline{\omega}^* = \frac{1}{\|\underline{u}_0 + \underline{u}\|}\left\{\begin{matrix} 0 \\ -\bar{q}^T\dot{\underline{u}} \\ \bar{p}^T\dot{\underline{u}} \end{matrix}\right\}. \quad (15)$$

As expected, when evaluated in the material basis, the first component of the angular velocity vector vanishes. Furthermore, the other to components are not fully defined because the orientations of unit vectors $\bar{p}$ and $\bar{q}$ in the normal plane are not defined. Hence, it is assumed that the rigid body is in the shape of a cylinder of mass m and its moment of inertia about unit vectors $\bar{p}$ or $\bar{q}$ is ϱ. With these restrictions, the kinetic energy of the rigid body reduces to

$$K = \frac{1}{2}\left(m\|\underline{v}_M\|^2 + \frac{\varrho}{\|\underline{u}_0 + \underline{u}\|^2}\left[\|\dot{\underline{u}}\|^2 - (\bar{e}^T\dot{\underline{u}})^2\right]\right). \quad (16)$$

References

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[2] W.A. Welsh. Simulation and correlation of a helicopter air-oil strut dynamic response. In *American Helicopter Society 43rd Annual Forum Proceedings*, St. Louis, Missouri, May 18-20 1987.

[3] W.H. Press, B.P. Flannery, S.A. Teutolsky, and W.T. Vetterling. *Numerical Recipes. The Art of Scientific Computing.* Cambridge University Press, Cambridge, 1990.