Dymore User's Manual

Representation of rotations and orientations

Contents

1 Introduction

In the present multibody formulation, orientations and rotations are represented with the help of the Wiener-Milenković parameters defined in section 2. It is sometimes convenient, however, to define orientations and rotations by different means, such as the Euler angles defined in section 3. Four different Euler angle representations are used here.

1. Euler angles with *3-1-3 sequence*, see section 3.1,

2. Euler angles with *3-2-3 sequence*, see section 3.2,

3. Euler angles with *3-2-1 sequence*, see section 3.3,

4. Euler angles with *3-1-2 sequence*, see section 3.4.

2 Wiener-Milenković parameters

At all the nodes of the model, six degrees of freedom are used: three displacements and three rotations. Both displacements and rotations are measured in the inertial frame. Rotations are represented with the **Wiener-Milenković parameters** [1, 2], also called **conformal rotation vector**, defined as

$$\underline{c} = 4 \tan \frac{\phi}{4} \, \underline{n}, \tag{1}$$

where ϕ is the magnitude of the rotation and $\underline{n}$ the unit vector about which it takes place. This representation is a direct consequence of Euler's theorem on rotations states that *any arbitrary finite rotation that leaves a point fixed can be viewed as a single rotation of magnitude* ϕ *about a unit vector* $\underline{n}$.

2.1 The rotation tensor

The rotation tensor can be expressed in terms of Wiener-Milenković parameters as

$$R(\underline{c}) = \frac{1}{(4-c_0)^2}\left[\begin{array}{ccc} c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) \\ 2(c_1c_2+c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) \\ 2(c_1c_3-c_0c_2) & 2(c_2c_3+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2 \end{array}\right]. \tag{2}$$

where

$$c_0 = 2(1-\tan^2\frac{\phi}{4}) = 2 - \frac{1}{8}\underline{c}^T\underline{c}. \tag{3}$$

2.2 The angular velocity vector

The relationship between the angular velocity vector, $\underline{\omega}$, and derivatives of the Wiener-Milenković parameters, $\underline{\dot{c}}$ is $\underline{\omega} = \underline{\underline{H}}\,\underline{\dot{c}}$, where the tangent tensor, $\underline{\underline{H}}$, is defined as

$$\underline{\underline{H}}(\underline{c}) = \frac{2}{(4-c_0)^2}\left[\begin{array}{ccc} c_0+c_1c_1/4 & c_1c_2/4-c_3 & c_1c_3/4+c_2 \\ c_1c_2/4+c_3 & c_0+c_2c_2/4 & c_2c_3/4-c_1 \\ c_1c_3/4-c_2 & c_2c_3/4+c_1 & c_0+c_3c_3/4 \end{array}\right]. \tag{4}$$

2.3 Composition of rotations

Let $\underline{p}$ and $\underline{q}$ be the Wiener-Milenković parameters of two successive rotations, and $\underline{r}$ the conformal rotation parameters of the composed rotation, such that $R(\underline{r}) = R(\underline{p})R(\underline{q})$. The composition formulæ are

$$r_0 = \frac{4}{\Delta_1}\left(p_0q_0 - \underline{p}^T\underline{q}\right); \tag{5}$$

$$\underline{r} = \frac{4}{\Delta_1}\left(q_0\,\underline{p} + p_0\,\underline{q} + \tilde{p}\,\underline{q}\right) \tag{6}$$

where $\Delta_1 = (4-p_0)(4-q_0) + (p_0q_0 - \underline{p}^T\underline{q})$.

2.4 Extended Wiener-Milenković parameters

This representation is limited to rotation angles $-2\pi < \phi < 2\pi$, since it presents a singularity $\underline{c} \to \infty$ when $|\phi| \to 2\pi$. The range of validity of the Wiener-Milenković parameterization can be *extended* by using a *rescaling operation*. This operation is based on the observation that rotations of magnitudes ϕ and $\phi' = \phi \pm 2\pi$ about the same axis $\underline{n}$ correspond to the same final configuration. The norm of the Wiener-Milenković parameters $\|\underline{p}\| = p \leq 4$ when $|\phi| \leq \pi$. Let $\underline{p}$ and $\underline{p}'$ be associated with the rotations ϕ and ϕ', respectively. The relationship between these two sets of parameters is

$$\underline{p}' = 4\underline{n}\,\tan\phi'/4 = 4\underline{n}\,\tan(\phi/4 \pm \pi/2) = -4\underline{n}\,1/\tan\phi/4 = -\underline{p}/(\tan^2\phi/4), \tag{7}$$

which writes

$$\underline{p}' = -\frac{\nu}{1-\nu}\,\underline{p}, \tag{8}$$

where $\nu = 2/(4 - c_0)$. Taking the norm of eq. (7) yields $p' = p/\tan^2 \phi/4$, and hence, $pp' = p^2 \tan^2 \phi/4$, or

$$p\, p' = 16. \tag{9}$$

If $\pi < |\phi| < 2\pi$, $p > 4$, and hence $p' < 4$; in other words, the rescaling operation decreases the norm of the vector parameterization. Let $\underline{p}_1$, $\underline{p}_2$, and $\underline{p}$ be the parameters of three rotation tensors such that $R(\underline{p}) = R(\underline{p}_1)R(\underline{p}_2)$. The composition formula, eq.(6), is then used to obtain the following update relationship

$$\underline{p} = \frac{\nu_1 \nu_2}{\nu} \left(\frac{1}{\varepsilon_2} \underline{p}_1 + \frac{1}{\varepsilon_1} \underline{p}_2 + \frac{1}{2} \tilde{p}_1 \underline{p}_2 \right), \tag{10}$$

where, in view of eq. (5), $2\nu - 1 = \cos \phi/2 = \nu_1 \nu_2 \left(1/\varepsilon_1 \varepsilon_2 - \underline{p}_1^T \underline{p}_2/4\right)$. As incremental rotations are added to the initial orientation, p increases and when $|\phi|$ becomes larger than π, $p > 4$ and a rescaling operation, eq. (8), becomes necessary. The two operations, update and rescaling, are conveniently combined into a single operation as follows

$$\underline{p} = \begin{cases} \dfrac{\nu_1 \nu_2}{\nu} \left(\dfrac{1}{\varepsilon_2} \underline{p}_1 + \dfrac{1}{\varepsilon_1} \underline{p}_2 + \dfrac{1}{2} \tilde{p}_1 \underline{p}_2 \right) & \text{if} \quad \nu \geq \dfrac{1}{2} \\ -\dfrac{\nu_1 \nu_2}{1 - \nu} \left(\dfrac{1}{\varepsilon_2} \underline{p}_1 + \dfrac{1}{\varepsilon_1} \underline{p}_2 + \dfrac{1}{2} \tilde{p}_1 \underline{p}_2 \right) & \text{if} \quad \nu \leq \dfrac{1}{2} \end{cases} \tag{11}$$

2.5 Examples

It is important to note that *the Wiener-Milenković parameters are not rotation angles*. Even in the case of a planar, the Wiener-Milenković parameters do not represent rotation angles. Indeed, let $\phi = \pi = 3.14$, $\underline{n}^T = \lfloor 1,\ 0,\ 0 \rfloor$, the corresponding parameters are $\underline{c} = 4 \tan \phi/4 \lfloor 1,\ 0,\ 0 \rfloor = 4 \lfloor 1,\ 0,\ 0 \rfloor$.

Fig. 1 shows a time dependent planar rotation $\phi = -20t$. Note that although ϕ is a linear function of time, $c = 4 \tan(-\phi/4) = -4 \tan 5t$ is not. When ϕ reaches a value of $-\pi$, *i.e.* when $-\pi = -20t$ or $t = 0.157$, a rescaling operation takes place. Note that the Wiener-Milenković parameters present discontinuities at each rescaling operation, however, the rotation tensor, see eq. (2), or angular velocity, see eq. (4), will not.

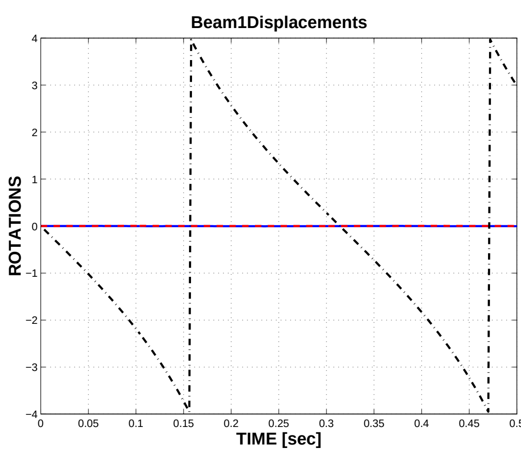

Figure 1: Planar rotation represented with Wiener-Milenković parameters.

3 Euler angles

An arbitrary rotation from $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ to $\mathcal{E} = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$ can be viewed as a succession of three planar rotations about three different axes [3].

Figure 2 shows one possible set of three planar rotations, which can be described as follows.

1. A planar rotation of magnitude ϕ, called *precession*, about axis $\bar{\imath}_3$ brings basis $\mathcal{I}$ to basis $\mathcal{A} = (\bar{a}_1, \bar{a}_2, \bar{a}_3)$. The corresponding direction cosine matrix is

$$\left\{ \begin{array}{rrrr} \bar{a}_1 = & \cos\phi\ \bar{\imath}_1 & +\sin\phi\ \bar{\imath}_2, & \\ \bar{a}_2 = & -\sin\phi\ \bar{\imath}_1 & +\cos\phi\ \bar{\imath}_2, & \\ \bar{a}_3 = & & & \bar{\imath}_3. \end{array} \right. \iff \underline{\underline{D}}_3(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

2. A planar rotation of magnitude θ, called *nutation*, about axis $\bar{a}_1$ brings basis $\mathcal{A}$ to basis $\mathcal{B} = (\bar{b}_1, \bar{b}_2, \bar{b}_3)$. The corresponding direction cosine matrix is

$$\left\{ \begin{array}{rrrr} \bar{b}_1 = & \bar{a}_1, & & \\ \bar{b}_2 = & & \cos\theta\ \bar{a}_2 & +\sin\theta\ \bar{a}_3, \\ \bar{b}_3 = & & -\sin\theta\ \bar{a}_2 & +\cos\theta\ \bar{a}_3. \end{array} \right. \iff \underline{\underline{D}}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}. \tag{13}$$

3. A planar rotation of magnitude ψ, called *spin*, about axis $\bar{b}_3$ brings basis $\mathcal{B}$ to basis $\mathcal{E}$. Once again, the corresponding direction cosine matrix is

$$\left\{ \begin{array}{rrrr} \bar{e}_1 = & \cos\psi\ \bar{b}_1 & +\sin\psi\ \bar{b}_2, & \\ \bar{e}_2 = & -\sin\psi\ \bar{b}_1 & +\cos\psi\ \bar{b}_2, & \\ \bar{e}_3 = & & & \bar{b}_3. \end{array} \right. \iff \underline{\underline{D}}_3(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{14}$$

The relationship between bases $\mathcal{I}$ and $\mathcal{E}$ is obtained by combining the three successive rotations described by eqs. (12) to (14) to find

$$\left\{ \begin{array}{rlll} \bar{e}_1 = & (\ \ C_\phi C_\psi - S_\phi C_\theta S_\psi)\ \bar{\imath}_1 & +(\ \ S_\phi C_\psi + C_\phi C_\theta S_\psi)\ \bar{\imath}_2 & +S_\theta S_\psi\ \bar{\imath}_3, \\ \bar{e}_2 = & (-C_\phi S_\psi - S_\phi C_\theta C_\psi)\ \bar{\imath}_1 & +(-S_\phi S_\psi + C_\phi C_\theta C_\psi)\ \bar{\imath}_2 & +S_\theta C_\psi\ \bar{\imath}_3, \\ \bar{e}_3 = & S_\phi S_\theta\ \bar{\imath}_1 & -C_\phi S_\theta\ \bar{\imath}_2 & +C_\theta\ \bar{\imath}_3, \end{array} \right. \tag{15}$$

where the following short-hand notations were used: $C_\phi = \cos\phi$, $S_\phi = \sin\phi$, etc.

The three rotation angles, ϕ, θ, and ψ, are called the *Euler angles*. The direction cosine matrix expressed in terms of Euler angles becomes

$$\underline{\underline{D}}_{\mathit{3\text{-}1\text{-}3}} = \begin{bmatrix} C_\phi C_\psi - S_\phi C_\theta S_\psi & -C_\phi S_\psi - S_\phi C_\theta C_\psi & S_\phi S_\theta \\ S_\phi C_\psi + C_\phi C_\theta S_\psi & -S_\phi S_\psi + C_\phi C_\theta C_\psi & -C_\phi S_\theta \\ S_\theta S_\psi & S_\theta C_\psi & C_\theta \end{bmatrix}. \tag{16}$$

It is often important to perform the inverse operation: given a direction cosine matrix, find the corresponding Euler angles. The following process will yield the desired angles. Assuming $D_{32} \neq 0$,

$$\tan\psi = D_{31}/D_{32}, \tag{17a}$$

$$\sin\theta = D_{31}\sin\psi + D_{32}\cos\psi, \quad \cos\theta = D_{33}, \tag{17b}$$

$$\sin\phi = D_{21}\cos\psi - D_{22}\sin\psi, \quad \cos\phi = D_{11}\cos\psi - D_{12}\sin\psi. \tag{17c}$$

To remove the ambiguity associated with inverse trigonometric functions, both sine and cosines of the angles are derived, leading to a definition of each angle in the range $[-\pi, +\pi]$.[1]

When $\theta = 0$ or π, a singularity occurs. In fact, the process then reduces to a single rotation of magnitude $(\phi + \psi)$ or $(\phi - \psi)$ for $\theta = 0$ or π, respectively, because the direction cosine matrix reduces to

$$\underline{\underline{D}} = \begin{bmatrix} \cos(\phi \pm \psi) & -\sin(\phi \pm \psi) & 0 \\ \sin(\phi \pm \psi) & \cos(\phi \pm \psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{18}$$

[1] In computer implementations, these operations are conveniently performed with the help of the function $\text{atan2}(y, x) = \tan^{-1}(y/x)$, yielding an angle in the range $[-\pi, +\pi]$.

Clearly, angles ϕ and ψ cannot be determined individually, the sole combination $\phi \pm \psi$ can be evaluated.

The Euler angles introduced above correspond to the following sequence of planar rotations: a rotation of magnitude ϕ, about axis $\bar{\imath}_3$, then, a rotation of magnitude θ about axis $\bar{a}_1$, and finally, a rotation of magnitude ψ about axis $\bar{b}_3$. This sequence will be called the "*3-1-3* sequence" to indicate the sequence of body axes about which the three successive rotations are taking place.

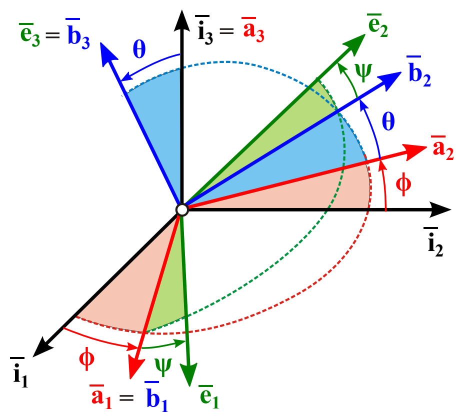

Figure 2: An arbitrary rotation viewed as three successive planar rotations.

Clearly, Euler angles could be defined in several different manners: the first rotation could occur about either of the three axes, $\bar{\imath}_1$, $\bar{\imath}_2$, or $\bar{\imath}_3$, offering three choices. Because two consecutive rotations cannot take place about the same axis, two alternatives are possible for the second rotation. Two choices are again possible for the last rotation.

In all, $3\times2\times2 = 12$ possible choices exist, corresponding to sequences labeled *1-2-1*, *1-2-3*, *1-3-1*, *1-3-2*, *2-1-2*, *2-1-3*, *2-3-1*, *2-3-2*, *3-1-2*, *3-1-3*, *3-2-1* and *3-2-3*. Three of these sequences, *3-2-3*, *3-2-1* and *3-1-2* will be the focus of problems below.

The representation of rotation in terms of three Euler angles shows that the direction cosine matrix can be expressed in terms of three parameters only. This representation, however, presents several drawbacks. First, Euler angles can be defined in several different manners, and the choice of the rotation sequence is entirely arbitrary. Furthermore, the expression for the direction cosine matrix, as seen for this example in eq. (16), is rather complicated and involves the evaluation of numerous trigonometric functions. Finally, singularities will occur in the evaluation of Euler angles from a direction cosine matrix for all 12 possible sequences.

The detailed description of Euler angle with four different sequences appears in the following sections.

3.1 Euler angles with 3-1-3 sequence

The orientation of a triad is defined by three Euler angles, measured in degrees, using the *3-1-3 sequence*. Euler angles, both at input or output are **measured in degrees**. This defines three consecutive planar rotations:

- A planar rotation of magnitude ϕ_1, called *precession*, about axis $\bar{\imath}_3$ brings $\mathcal{I}$ to $\mathcal{A}$.
- A planar rotation of magnitude ϕ_2, called *nutation*, about axis $\bar{a}_1$ brings $\mathcal{A}$ to $\mathcal{B}$.
- A planar rotation of magnitude ϕ_3, called *spin*, about axis $\bar{b}_3$ brings $\mathcal{B}$ to $\mathcal{E}$.

The rotation matrix is then

$$D_{3-1-3} = \begin{bmatrix} C_1C_3 - S_1C_2S_3 & -C_1S_3 - S_1C_2C_3 & S_1S_2 \\ S_1C_3 + C_1C_2S_3 & -S_1S_3 + C_1C_2C_3 & -C_1S_2 \\ S_2S_3 & S_2C_3 & C_2 \end{bmatrix}. \tag{19}$$

When requesting a rotation output by means of a sensor, it is necessary to perform the inverse operation: given a direction cosine matrix, find the corresponding Euler angles. The following process will yield the desired angles.

$$\phi_3 = \text{atan2}(D_{31}, D_{32}); \quad \text{if} \quad D_{32} \neq 0; \tag{20a}$$
$$\phi_2 = \text{atan2}(D_{31}S_3 + D_{32}C_3, D_{33}); \tag{20b}$$
$$\phi_1 = \text{atan2}(D_{21}C_3 - D_{22}S_3, D_{11}C_3 - D_{12}S_3). \tag{20c}$$

It is clear that when $\theta = 0$ or π, a singularity occurs.

3.2 Euler angles with 3-2-3 sequence

The orientation of a triad is defined by three Euler angles, measured in degrees, using the *3-2-3 sequence*. Euler angles, both at input or output are **measured in degrees**. This defines three consecutive planar rotations:

- A planar rotation of magnitude ϕ_1, called *precession*, about axis $\bar{\imath}_3$ brings $\mathcal{I}$ to $\mathcal{A}$.
- A planar rotation of magnitude ϕ_2, called *nutation*, about axis $\bar{a}_2$ brings $\mathcal{A}$ to $\mathcal{B}$.
- A planar rotation of magnitude ϕ_3, called *spin*, about axis $\bar{b}_3$ brings $\mathcal{B}$ to $\mathcal{E}$.

The rotation matrix is then

$$D_{\mathit{3-2-3}} = \begin{bmatrix} C_1C_2C_3 - S_1S_3 & -C_1C_2S_3 - S_1C_3 & C_1S_2 \\ S_1C_2C_3 + C_1S_3 & -S_1C_2S_3 + C_1C_3 & S_1S_2 \\ -S_2C_3 & S_2S_3 & C_2 \end{bmatrix}. \tag{21}$$

When requesting a rotation output by means of a sensor, it is necessary to perform the inverse operation: given a direction cosine matrix, find the corresponding Euler angles. The following process will yield the desired angles.

$$\phi_3 = \text{atan2}(D_{32}, -D_{31}); \quad \text{if} \quad D_{31} \neq 0; \tag{22a}$$
$$\phi_2 = \text{atan2}(D_{32}S_3 - D_{31}C_3, D_{33}); \tag{22b}$$
$$\phi_1 = \text{atan2}(D_{21}C_3 - D_{22}S_3, D_{11}C_3 - D_{12}S_3). \tag{22c}$$

It is clear that when $\theta = 0$ or π, a singularity occurs.

3.3 Euler angles with 3-2-1 sequence

The orientation of a triad is defined by three Euler angles, measured in degrees, using the *3-2-1 sequence* which is popular for airplane flight mechanics. Euler angles, both at input or output are **measured in degrees**. This defines three consecutive planar rotations:

- A planar rotation of magnitude ϕ_1, called *heading*, about axis $\bar{\imath}_3$ brings $\mathcal{I}$ to $\mathcal{A}$.
- A planar rotation of magnitude ϕ_2, called *attitude*, about axis $\bar{a}_2$ brings $\mathcal{A}$ to $\mathcal{B}$.
- A planar rotation of magnitude ϕ_3, called *bank*, about axis $\bar{b}_1$ brings $\mathcal{B}$ to $\mathcal{E}$.

The rotation matrix is then

$$D_{\mathit{3-2-1}} = \begin{bmatrix} C_1C_2 & -S_1C_3 + C_1S_2S_3 & S_1S_3 + C_1S_2C_3 \\ S_1C_2 & C_1C_3 + S_1S_2S_3 & -C_1S_3 + S_1S_2C_3 \\ -S_2 & C_2S_3 & C_2C_3 \end{bmatrix} \tag{23}$$

When requesting a rotation output by means of a sensor, it is necessary to perform the inverse operation: given a direction cosine matrix, find the corresponding Euler angles. The following process will yield the desired angles.

$$\phi_3 = \text{atan2}(D_{32}, D_{33}); \quad \text{if} \quad D_{33} \neq 0; \tag{24a}$$
$$\phi_2 = \text{atan2}(-D_{31}, D_{32}S_3 + D_{33}C_3); \tag{24b}$$
$$\phi_1 = \text{atan2}(D_{21}, D_{11}). \tag{24c}$$

It is clear that when $\theta = \pm\pi/2$, a singularity occurs.

3.4 Euler angles with 3-1-2 sequence

The orientation of a triad is defined by three Euler angles, measured in degrees, using the *3-1-2 sequence*. Euler angles, both at input or output are **measured in degrees**. This defines three consecutive planar rotations:

- A planar rotation of magnitude ϕ_1, about axis $\bar{\imath}_3$ brings $\mathcal{I}$ to $\mathcal{A}$.
- A planar rotation of magnitude ϕ_2 about axis $\bar{a}_1$ brings $\mathcal{A}$ to $\mathcal{B}$.
- A planar rotation of magnitude ϕ_3 about axis $\bar{b}_2$ brings $\mathcal{B}$ to $\mathcal{E}$.

The rotation matrix is then

$$D_{3-1-2} = \begin{bmatrix} C_1C_3 - S_1S_2S_3 & -S_1C_2 & C_1S_3 + S_1S_2C_3 \\ S_1C_3 + C_1S_2S_3 & C_1C_2 & S_1S_3 - C_1S_2C_3 \\ -C_2S_3 & S_2 & C_2C_3 \end{bmatrix} \tag{25}$$

When requesting a rotation output by means of a sensor, it is necessary to perform the inverse operation: given a direction cosine matrix, find the corresponding Euler angles. The following process will yield the desired angles.

$$\phi_3 = \text{atan2}(-D_{31}, D_{33}); \quad \text{if} \quad D_{33} \neq 0; \tag{26a}$$
$$\phi_2 = \text{atan2}(D_{32}, D_{33}C_3 - D_{31}S_3); \tag{26b}$$
$$\phi_1 = \text{atan2}(-D_{12}, D_{22}). \tag{26c}$$

It is clear that when $\theta = \pm\pi/2$, a singularity occurs.

References

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