Dymore User's Manual
Fourier Analysis

1 Discrete Fourier transform

Let $s(t)$ denote a time dependent signal. The *discrete Fourier transform* of the signal is an approximation of the following form

$$s(t) \approx a_0 + \sum_{n=1}^{N_f} \left(a_n^s \sin n\Omega t + a_n^c \cos n\Omega t \right). \tag{1}$$

This analysis depends on *two input parameters*, the number of frequencies, N_f, to be used in the expansion, and the base circular frequency of the signal, Ω. The coefficients of the discrete Fourier expansion are computed using the following formulæ

$$a_0 = \frac{1}{T} \int_0^T s(t) \; \mathrm{d}t, \tag{2a}$$

$$a_n^s = \frac{2}{T} \int_0^T s(t) \sin n\Omega t \; \mathrm{d}t, \tag{2b}$$

$$a_n^c = \frac{2}{T} \int_0^T s(t) \cos n\Omega t \; \mathrm{d}t. \tag{2c}$$

It is assumed that the signal is defined over one period, T, where $T = 2\pi/\Omega$. If it is not, signal preconditioning must be used to select one period of the signal precisely. The integrals appearing in eqs. (2) are evaluated using Simpson's rule.

2 Fourier transform

Consider now a signal defined by a sequence of discrete pairs, (t_i, s_i), $i = 1, 2, \ldots, N$, where $s_i = s(t_i)$ is the discrete value of the signal at time t_i, and N the number of sampling points. Note that the sampling points are not necessarily equally spaced in time.

The Fourier transform algorithm computes the components S_k given the discrete values of the signal, s_n,

$$S_k = \frac{2}{n_s} \sum_{n=0}^{n_s-1} s_n e^{-\frac{2\pi i}{n_s} nk}, \quad k = 0, 1, \ldots N_f - 1. \tag{3}$$

Two parameters appear in this formula: the number of time steps of the signal, n_s, and the number of frequencies, N_f, to be extracted.

If the the number of time steps, n_s, is not defined, the default value is $n_s = N$. Since the algorithms used for the Fourier transform assume equally spaced sampling points, the first phase of the algorithm is to resample the original signal, defined by N discrete values, to create a new signal with n_s equally spaced discrete values. Linear interpolation is used in this operation. While the Fourier transform of eq. (3) is defined for all values of $k = 1, 2, \ldots, n_s$, only those values below the Nyquist frequency are relevant, *i.e.* $k \leq n_s/2$. If N_f is not defined, $N_f = n_s/2$.

3 On-line Fourier analysis

The goal of the on-line Fourier analysis is to estimate the magnitudes of the Fourier components of a signal at specific, given frequencies. Note that the frequency content of the signal must be known ahead of time to perform an on-line Fourier analysis, which solely estimates the amplitudes of the signal at those frequencies. If the user provided frequency content, f_k, $k = 1, 2 \ldots, N_f$, is not correct, erroneous amplitudes will be estimated.

The algorithm proceeds as follows. Assume that estimates of the signal's Fourier component magnitudes are available and let them be denoted a_0, a_k^s, $k = 1, 2, \ldots, N_f$ and a_k^c, $k = 1, 2, \ldots, N_f$. The value of the signal at time $t = t_f$, denoted $\hat{s}_f$, is then readily estimated as

$$\hat{s}_f = a_0 + \sum_{k=1}^{N_f} \left(a_k^s \sin 2\pi f_k t_f + a_k^c \cos 2\pi f_k t_f \right), \tag{4}$$

where N_f is the number of frequencies to be used in the Fourier expansion, and f_k, $k = 1, 2 \ldots, N_f$, the frequency content of the signal.

The actual value of the signal at this time is s_f, and the corresponding error in the estimation is $e_f = s_f - \hat{s}_f$. This error is then used to update the estimates of the signal's Fourier component magnitudes using the following formulæ

$$\bar{a}_0 = a_0 + g e_f, \tag{5a}$$

$$\bar{a}_k^s = a_k^s + g e_f \sin 2\pi f_k t, \quad k = 1, 2, \ldots, N_f, \tag{5b}$$

$$\bar{a}_k^c = a_k^c + g e_f \cos 2\pi f_k t, \quad k = 1, 2, \ldots, N_f, \tag{5c}$$

where $\bar{a}_0$, $\bar{a}_k^s$, $k = 1, 2, \ldots, N_f$, and $\bar{a}_k^c$, $k = 1, 2, \ldots, N_f$ are the updated estimates of the Fourier components, and g the gain coefficient. As a new pair (t_f, s_f) becomes available, the estimates of all Fourier components are refined; if the input signal is periodic and consists of the defined harmonics, convergence is rapid. The optimal value of the gain coefficient is to twice the time step size divided by the fundamental period of the signal.