Dymore User's Manual

Formulation and finite element implementation of beam elements

Contents

1 Formulation of beam equations

A beam is defined as a structure having one of its dimensions much larger than the other two. The axis of the beam is defined along that longer dimension and its cross-section is normal to this axis. The cross-section's geometric and physical properties are assumed to vary smoothly along the beam's span. Civil engineering structures often consist of assemblies or grids of beams with cross-sections having shapes such as T's or I's. A large number of machine parts also are beam-like structures: linkages, transmission shafts, robotic arms, etc. Aeronautical structures such as aircraft wings or helicopter rotor blades are often treated as thin-walled beams. Finally, both tower and rotor blades of wind turbines also fall within the category of beams structures.

The solid mechanics theory of beams, more commonly referred to simply as "beam theory," plays an important role in structural analysis because it provides designers with simple tools to analyze numerous structures [1]. Within the framework of multibody dynamics, the governing equations for beam structures are nonlinear partial differential equations, and the finite element method is often used to obtain approximate numerical solutions of these equations.

Of course, the same finite element approach could also be used to model the same structures based on plate and shell, or even three-dimensional elasticity models, but at a much higher computation cost. Beam models are often used at a pre-design stage because they provide valuable insight into the behavior of structures.

Several beam theories have been developed based on various assumptions, and lead to different levels of accuracy. One of the simplest and most useful of these theories is due to Euler who analyzed the elastic deformation of a slender beam, a problem known as Euler's Elastica [2]. Euler-Bernoulli beam theory [1] is now commonly used in many civil, mechanical and aerospace applications, although shear deformable beam theories [3, 4], often called "Timoshenko beams," have also found wide acceptance. Reissner investigated beam theory for large strains [5] and large displacements of spatially curved members [6, 7].

In this section, the *geometrically exact beam theory* will be presented. The kinematic description of the problem developed in section 1.1 accounts for arbitrarily large displacements and rotation, hence the term "geometrically exact," although the strain components are assumed to remain small. The kinematics of geometrically beams was first presented by Simo *et al.* [8, 9], but similar developments were proposed by Borri and Merlini [10] or Danielson and Hodges [11, 12].

In many applications, however, beams are, in fact, complex build-up structures with solid or thin-walled cross-sections. In aeronautical constructions, for instance, the increasing use of laminated composite materials leads to heterogeneous, highly anisotropic structures. The analysis of complex cross-sections featuring composite materials and the determination of the associated sectional properties was first presented by Giavotto *et al.* [13, 14]. Their approach, based on linear elasticity theory, leads to a two-dimensional analysis of the beam's cross-section using finite elements, which yields the sectional stiffness characteristics in the form of a 6×6 stiffness matrix relating the six sectional deformations, three strains and three curvatures, to the sectional loads, three forces and three moments. Furthermore, the three-dimensional strain field at all points of the cross-section can be recovered once the sectional strains are known.

For nonlinear problems, the decomposition of the beam problem into a linear, two-dimensional analysis over the cross-section, and a nonlinear, one-dimensional analysis along its span was first proposed by Berdichevsky [15]. Hodges [16] has reviewed many approaches to beam modeling; he points out that although the two-dimensional finite element analysis of the cross-section seems to be computationally expensive, it is, in fact, a preprocessing step that is performed once only.

A unified theory presenting both linear, two-dimensional analysis over the cross-section, and a nonlinear, one-dimensional analysis along the beam's span was further refined by Hodges and his co-workers [17, 18]. The nonlinear, one-dimensional analysis along the beam's span corresponds the geometrically exact beam theory developed earlier based on simplified kinematic assumptions. More sophisticated beam theories have been developed that account for Vlasov effects [19] or the trapeze effect [20]. Detailed developments of nonlinear composite beam theory developed by Hodges and his coworkers are found in his textbook [21] and applications to multibody systems in ref. [22].

1.1 Kinematics of the problem

Figure 1 depicts an initially curved and twisted beam of length L, with a cross-section of arbitrary shape and area $\mathcal{A}$. The volume of the beam is generated by sliding the cross-section along the reference line of the beam, which is defined by an arbitrary curve in space. Curvilinear coordinate α_1 defines the intrinsic parameterization of this curve, *i.e.*, it measures length along the beam's reference line. Point $\mathbf{B}$ is located at the intersection of the reference line with the plane of the cross-section.

In the reference configuration, an orthonormal basis, $\mathcal{B}_0(\alpha_1) = (\bar{b}_1, \bar{b}_2, \bar{b}_3)$, is defined at point $\mathbf{B}$. Vector $\bar{b}_1$ is the unit tangent vector to the reference curve at that point, and unit vectors $\bar{b}_2$ and $\bar{b}_3$ define the plane to the cross-section. An inertial reference frame, $\mathcal{F}^I = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$, is defined, and the components of the rotation tensor that brings basis $\mathcal{I}$ to $\mathcal{B}_0$, resolved in basis $\mathcal{I}$, are denoted $\underline{\underline{R}}_0(\alpha_1)$.

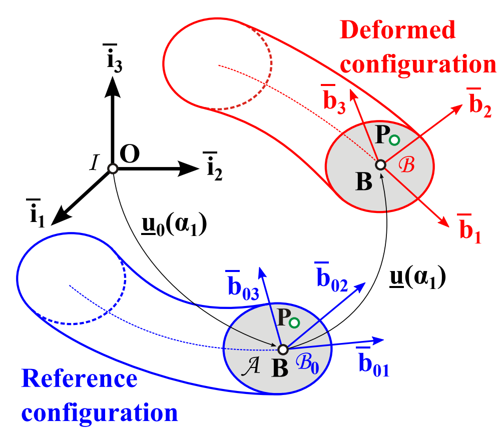

Figure 1: Curved beam in the reference and deformed configurations.

The position vector of point $\mathbf{B}$ along the beam's reference line is denoted $\underline{x}_0(\alpha_1)$. The position vector of material point $\mathbf{P}$ of the beam then becomes $\underline{x}(\alpha_1, \alpha_2, \alpha_3) = \underline{x}_0(\alpha_1) + \alpha_2\ \bar{b}_2 + \alpha_3\ \bar{b}_3$, where α_2 and α_3 are the material coordinates along unit vectors $\bar{b}_2$ and $\bar{b}_3$, respectively. Coordinates α_1, α_2, and α_3 form a natural choice of coordinates to represent the configuration of the beam.

1.1.1 The displacement field

In the deformed configuration, all the material points located on a cross-section of the beam move to new positions. This motion is decomposed into two parts, a rigid body motion and a warping displacement field. The rigid body motion consists of a translation of the cross-section, characterized by displacement vector $\underline{u}(\alpha_1)$ of reference point **B**, and of a rotation of the cross-section, which brings basis $\mathcal{B}_0$ to $\mathcal{B}(\alpha_1) = (\bar{B}_1, \bar{B}_2, \bar{B}_3)$, see fig. 1. The components of the rotation tensor that brings basis $\mathcal{B}_0$ to $\mathcal{B}$, resolved in basis $\mathcal{I}$, are denoted $\underline{\underline{R}}(\alpha_1)$.

The warping displacement field is defined as $\underline{w}(\alpha_1, \alpha_2, \alpha_3) = w_1 \bar{B}_1 + w_2 \bar{B}_2 + w_3 \bar{B}_3$. This displacement field represents a warping that includes both in-plane and out-of-plane deformations of the cross-section. To be uniquely defined, the warping field should be orthogonal to the rigid body motion [13, 21]. Consequently, unit vectors $\bar{B}_2$ and $\bar{B}_3$ define the average plane of the cross-section and vector $\bar{B}_1$ is orthogonal to that plane.

The position vector of point **P** in the deformed configuration now becomes

$$\underline{X}(\alpha_1, \alpha_2, \alpha_3) = \underline{X}_0 + w_1\ \bar{B}_1 + (w_2 + \alpha_2)\bar{B}_2 + (w_3 + \alpha_3)\bar{B}_3. \tag{1}$$

The position of point **B** is expressed as $\underline{X}_0(\alpha_1) = \underline{x}_0 + \underline{u}$. Because $\bar{B}_i = \underline{\underline{R}}\ \bar{b}_i = (\underline{\underline{R}}\,\underline{\underline{R}}_0)\ \bar{\imath}_i$, eq. (1) becomes

$$\underline{X}(\alpha_1, \alpha_2, \alpha_3) = \underline{x}_0 + \underline{u} + (\underline{\underline{R}}\,\underline{\underline{R}}_0)\,(\underline{w} + \alpha_2\ \bar{\imath}_2 + \alpha_3\ \bar{\imath}_3)\,. \tag{2}$$

The warping displacement field is computed from the geometric and stiffness properties of the cross-section, typically by solving a two-dimensional finite element problem over the cross-section, as described in refs. [13, 21].

1.1.2 The sectional strain measures

The sectional strain measures for beams with shallow curvature are defined as

$$\underline{\mathcal{e}} = \begin{Bmatrix} \underline{\epsilon} \\ \underline{\kappa} \end{Bmatrix} = \begin{Bmatrix} \underline{x}_0' + \underline{u}' - (\underline{\underline{R}}\,\underline{\underline{R}}_0)\ \bar{\imath}_1, \\ \underline{k} + \underline{\underline{R}}\,\underline{k}_i \end{Bmatrix}, \tag{3}$$

where $\underline{k} = \mathrm{axial}(\underline{\underline{R}}'\underline{\underline{R}}^T)$ are the components of the sectional curvature vector resolved in the inertial basis and $\underline{k}_i$ the components of the corresponding curvature vector in the reference configuration. Notation $(\cdot)'$ indicates a derivative with respect to α_1. The strain components resolved in the convected material basis, $\mathcal{B}$, are denoted $\underline{\epsilon}^* = (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T\underline{\epsilon}$ and consist of the sectional axial and shear strains. The curvature components resolved in the same material basis are denoted $\underline{\kappa}^* = (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T\underline{\kappa}$ and consist of the sectional twisting and bending curvatures. Notation $(\cdot)^*$ indicates the components of vectors and tensors resolved in the material basis.

By definition, a rigid body motion is a motion that generates no strains. This implies that the following rigid body motion, $\underline{u}(\alpha_1) = \underline{u}^R + (\underline{\underline{R}}^R - \underline{\underline{I}})\underline{x}_0(\alpha_1)$, $\underline{\underline{R}}(\alpha_1) = \underline{\underline{R}}^R$, consisting of a translation, $\underline{u}^R$, and a rotation about the origin characterized by a rotation matrix, $\underline{\underline{R}}^R$, should generate no straining of the beam. It can be readily verified with the help of eqs. (3) that such rigid body motion results in $\underline{\epsilon} = 0$ and $\underline{\kappa} = 0$, as expected.

1.2 Governing equations

For the problem at hand, the principle of virtual work states

$$\int_0^L (\delta\underline{\epsilon}^{*T}\underline{N}^* + \delta\underline{\kappa}^{*T}\underline{M}^*)\,\mathrm{d}\alpha_1 = \delta W_{\mathrm{ext}}, \tag{4}$$

where $\underline{N}^*$ and $\underline{M}^*$ are the beam's sectional forces and moments, respectively. The sectional constitutive law relates the sectional strain measures to the sectional loads,

$$\begin{Bmatrix} \underline{N}^* \\ \underline{M}^* \end{Bmatrix} = \underline{\underline{\mathcal{C}}}^* \begin{Bmatrix} \underline{\epsilon}^* \\ \underline{\kappa}^* \end{Bmatrix}, \tag{5}$$

where $\underline{\underline{\mathcal{C}}}^*$ is the beam's 6×6 sectional stiffness matrix. This matrix is a byproduct of a two-dimensional finite element analysis over the beam's cross-section, as discussed in refs. [13, 21]. For homogeneous sections of simple geometry, exact or approximate analytical expressions are available for the stiffness matrix.

Variations in strain components are expressed using eq. (3) to find

$$\delta\underline{\epsilon}^* = (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T\left[\delta\underline{u}' + (\widetilde{x}_0' + \widetilde{u}')\underline{\delta\psi}\right], \tag{6a}$$

$$\delta\underline{\kappa}^* = (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T\underline{\delta\psi}'. \tag{6b}$$

where $\underline{\delta\psi} = \mathrm{axial}(\delta\underline{\underline{R}}\,\underline{\underline{R}}^T)$ is the virtual rotation vector. The principle of virtual work, eq. (4), now becomes

$$\int_0^L \left\{\left[\delta\underline{u}'^T + \underline{\delta\psi}^T(\widetilde{x}_0' + \widetilde{u}')^T\right]\underline{N} + \underline{\delta\psi}'^T\underline{M}\right\}\ \mathrm{d}\alpha_1 = \delta W_{\mathrm{ext}}, \tag{7}$$

where $\underline{N} = (\underline{\underline{R}}\,\underline{\underline{R}}_0)\underline{N}^*$ and $\underline{M} = (\underline{\underline{R}}\,\underline{\underline{R}}_0)\underline{M}^*$ are the beam's internal forces and moments, respectively, resolved in the inertial basis.

The virtual work done by the externally applied forces is expressed as $\delta W_{\text{ext}} = \int_0^L [\delta\underline{u}^T\underline{f} + \underline{\delta\psi}^T\underline{m}]\ \mathrm{d}\alpha_1$, where $\underline{f}$ and $\underline{m}$ denote the externally applied forces and moments per unit span of the beam, respectively.

The governing equations of the static problem then follow as

$$\underline{N}' = -\underline{f}, \tag{8a}$$

$$\underline{M}' + (\widetilde{x}_0' + \widetilde{u}')\underline{N} = -\underline{m}. \tag{8b}$$

1.3 Extension to dynamic problems

The developments presented thus far have focused on static problems. The inertial velocity vector, $\underline{v}$, of a material point is found by taking a time derivative of its inertial position vector, eq. (2), to find

$$\underline{v} = \underline{\dot{u}} + \underline{\underline{\dot{R}}}\,\underline{\underline{R}}_0\underline{s}^* = \underline{\dot{u}} + (\underline{\underline{R}}\,\underline{\underline{R}}_0)\widetilde{\omega}^*\underline{s}^* = \underline{\dot{u}} + (\underline{\underline{R}}\,\underline{\underline{R}}_0)\widetilde{s}^{*T}\underline{\omega}^*, \tag{9}$$

where contributions of warping of the cross-section have been ignored and $\underline{s}^{*T} = \left\{0, \alpha_2, \alpha_3\right\}$. Notation $\dot{(\cdot)}$ indicates a derivative with respect to time and $\underline{\omega}^*$ are the components of the angular velocity vector in the material system, $\underline{\omega}^* = (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T\underline{\omega}$, where $\underline{\omega} = \text{axial}(\underline{\underline{\dot{R}}}\,\underline{\underline{R}})$.

The components of the inertial velocity vector of a material point resolved in the material frame now become

$$\underline{v}^* = (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T\underline{v} = (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T\underline{\dot{u}} + \widetilde{s}^{*T}\underline{\omega}^*. \tag{10}$$

The total inertial velocity of a material point has two components: the first term, $(\underline{\underline{R}}\,\underline{\underline{R}}_0)^T\underline{\dot{u}}$, due to the translation of the cross-section, and the second term, $\widetilde{s}^{*T}\underline{\omega}^*$, due to its rotation.

1.3.1 The kinetic energy

The kinetic energy, K, of the beam is

$$K = \frac{1}{2}\int_0^L \int_{\mathcal{A}} \rho\ \underline{v}^{*T}\underline{v}^*\ \mathrm{d}\mathcal{A}\mathrm{d}\alpha_1, \tag{11}$$

where ρ is the mass density of the material per unit volume of the reference configuration. Introducing eq. (10) for the inertial velocity yields

$$K = \frac{1}{2}\int_0^L \int_{\mathcal{A}} \rho \left[\underline{\dot{u}}^T(\underline{\underline{R}}\,\underline{\underline{R}}_0) + \underline{\omega}^{*T}\widetilde{s}^*\right]\left[(\underline{\underline{R}}\,\underline{\underline{R}}_0)^T\,\underline{\dot{u}} + \widetilde{s}^{*T}\underline{\omega}^*\right]\ \mathrm{d}\mathcal{A}\mathrm{d}\alpha_1. \tag{12}$$

The following sectional mass constants are defined

$$m = \int_{\mathcal{A}} \rho\ \mathrm{d}\mathcal{A}, \quad \underline{\eta}^* = \frac{1}{m}\int_{\mathcal{A}} \rho\,\underline{s}^*\ \mathrm{d}\mathcal{A}, \quad \underline{\underline{\varrho}}^* = \int_{\mathcal{A}} \rho\widetilde{s}^*\widetilde{s}^{*T}\ \mathrm{d}\mathcal{A}, \tag{13}$$

where m is the mass of the beam per unit span, $\underline{\eta}^*$ the components of the position vector of the sectional center of mass with respect to point **B**, see fig. 1, and $\underline{\underline{\varrho}}^*$ the components of the sectional tensor of inertia per unit span, all resolved in the material basis.

After integration over the beam's cross-section, the kinetic energy, eq. (12), becomes

$$\begin{aligned} K &= \frac{1}{2}\int_0^L \left[m\underline{\dot{u}}^T\underline{\dot{u}} + 2m\underline{\dot{u}}^T(\underline{\underline{R}}\,\underline{\underline{R}}_0)\,\widetilde{\eta}^{*T}\underline{\omega}^* + \underline{\omega}^{*T}\underline{\underline{\varrho}}^*\underline{\omega}^*\right]\ \mathrm{d}\alpha_1 \\ &= \frac{1}{2}\int_0^L \underline{\mathcal{V}}^{*T}\underline{\underline{\mathcal{M}}}^*\underline{\mathcal{V}}^*\ \mathrm{d}\alpha_1. \end{aligned}$$

To obtain the compact form expressed by the second equality, the sectional mass matrix of the cross-section, resolved in the material basis, is defined as

$$\underline{\underline{\mathcal{M}}}^* = \begin{bmatrix} m\underline{\underline{I}} & m\widetilde{\eta}^{*T} \\ m\widetilde{\eta}^* & \underline{\underline{\varrho}}^* \end{bmatrix}, \tag{14}$$

and the sectional velocities, also resolved in the material basis, are given by

$$\underline{\mathcal{V}}^* = \begin{Bmatrix} (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T\underline{\dot{u}} \\ \underline{\omega}^* \end{Bmatrix} = \begin{bmatrix} (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T & \underline{\underline{0}} \\ \underline{\underline{0}} & (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T \end{bmatrix}\begin{Bmatrix} \underline{\dot{u}} \\ \underline{\omega} \end{Bmatrix} = (\underline{\underline{\mathcal{R}}}\,\underline{\underline{\mathcal{R}}}_0)^T\underline{\mathcal{V}}. \tag{15}$$

In this expression, the sectional velocities resolved in the inertial system were defined as $\underline{\mathcal{V}}^T = \{\underline{\dot{u}}^T, \quad \underline{\omega}^T\}$ and the following notation was introduced

$$\underline{\underline{\mathcal{R}}}\,\underline{\underline{\mathcal{R}}}_0 = \begin{bmatrix} (\underline{\underline{R}}\,\underline{\underline{R}}_0) & \underline{\underline{0}} \\ \underline{\underline{0}} & (\underline{\underline{R}}\,\underline{\underline{R}}_0) \end{bmatrix}. \tag{16}$$

The components of the sectional linear and angular momenta resolved in the material system, denoted $\underline{h}^*$ and $\underline{g}^*$, respectively, are

$$\underline{\mathcal{P}}^* = \begin{Bmatrix} \underline{h}^* \\ \underline{g}^* \end{Bmatrix} = \underline{\underline{\mathcal{M}}}^* \underline{\mathcal{V}}^*. \tag{17}$$

1.3.2 The governing equations

Variation of the kinetic energy is $\delta K = \int_0^L \delta\underline{\mathcal{V}}^{*T} \underline{\underline{\mathcal{M}}}^* \underline{\mathcal{V}}^* \, \mathrm{d}\alpha_1$, where the variations in velocities are $\delta[\underline{\dot{u}}^T(\underline{\underline{R}}\,\underline{\underline{R}}_0)] = (\delta\underline{\dot{u}}^T + \underline{\delta\psi}^T \dot{\widetilde{u}}^T)(\underline{\underline{R}}\,\underline{\underline{R}}_0)$ and $\delta\underline{\omega}^{*T} = \underline{\delta\dot{\psi}}^T(\underline{\underline{R}}\,\underline{\underline{R}}_0)$. Introducing these variations in the expression for the kinetic energy yields

$$\delta K = \int_0^L \left[(\delta\underline{\dot{u}}^T + \underline{\delta\psi}^T \dot{\widetilde{u}}^T)(\underline{\underline{R}}\,\underline{\underline{R}}_0)\,\underline{h}^* + \underline{\delta\dot{\psi}}^T (\underline{\underline{R}}\,\underline{\underline{R}}_0)\,\underline{g}^* \right] \, \mathrm{d}\alpha_1,$$

The components of the sectional linear and angular momenta, denoted $\underline{h}$ and $\underline{g}$, respectively, resolved in the inertial system are

$$\underline{\mathcal{P}} = \begin{Bmatrix} \underline{h} \\ \underline{g} \end{Bmatrix} = (\underline{\underline{\mathcal{R}}}\,\underline{\underline{\mathcal{R}}}_0)\underline{\mathcal{P}}^*, \tag{18}$$

where $\underline{\mathcal{P}}^*$ are the corresponding quantities resolved in the material frame, see eq. (17). The variation in kinetic energy finally can be written as

$$\delta K = \int_0^L (\delta\underline{\dot{u}}^T\,\underline{h} + \underline{\delta\psi}^T\,\dot{\widetilde{u}}^T\,\underline{h} + \underline{\delta\dot{\psi}}^T\,\underline{g}) \, \mathrm{d}\alpha_1. \tag{19}$$

With the help of eqs. (7) and (19), the governing equations of motion of the problem are obtained from Hamilton's principle, which states that

$$\int_{t_i}^{t_f} \int_0^L \Big\{ (\delta\underline{\dot{u}}^T + \underline{\delta\psi}^T\,\dot{\widetilde{u}}^T)\underline{h} + \underline{\delta\dot{\psi}}\,\underline{g} - (\delta\underline{u}'^T + \underline{\delta\psi}^T\,\widetilde{E}_1^T)\underline{N} \\ -\underline{\delta\psi}'^T\underline{M} + \delta\underline{u}^T\underline{f} + \underline{\delta\psi}^T\underline{m} \Big\} \, \mathrm{d}\alpha_1 \mathrm{d}t = 0.$$

Integration by parts yields the equations of motion of the problem

$$\underline{\dot{h}} - \underline{N}' = \underline{f}, \tag{20a}$$

$$\underline{\dot{g}} + \dot{\widetilde{u}}\,\underline{h} - \underline{M}' - (\widetilde{x}_0' + \widetilde{u}')\underline{N} = \underline{m}. \tag{20b}$$

1.4 Effect of extension-twist coupling

The beam theory developed in the previous sections assumes all strain components to remain small. In some cases, however, the coupling between extension and twisting of the beam becomes an important factor. The associated potential energy is

$$A = \frac{1}{2} c_{\mathrm{et}} \kappa_1^{*2} \epsilon_1^*, \tag{21}$$

where κ_1^* is the blade's twist rate and ϵ_1^* its axial extension, both resolved in the material basis. The extension-twist stiffness coefficient, c_{et}, is simple the sum of the bending stiffnesses, *i.e.*,

$$c_{\mathrm{et}} = H_{22} + H_{33}. \tag{22}$$

Taking a variation of the extension-twist potential yields $\delta A = 1/2\; c_{\mathrm{et}}\kappa_1^{*2}\delta\epsilon_1^* + c_{\mathrm{et}}\kappa_1^*\epsilon_1^*\delta\kappa_1^*$. This expression can be recast in a more compact form as

$$\delta A = \delta\underline{\epsilon}^*\,\underline{N}_{\mathrm{et}}^* + \delta\underline{\kappa}^*\,\underline{M}_{\mathrm{et}}^*, \tag{23}$$

where the force and moment arrays associated with the extension-twist coupling effect were defined as

$$\underline{N}_{\mathrm{et}}^* = 1/2\; c_{\mathrm{et}}\kappa_1^{*2}\bar{\imath}_1, \tag{24a}$$

$$\underline{M}_{\mathrm{et}}^* = \quad c_{\mathrm{et}}\kappa_1^*\epsilon_1^*\bar{\imath}_1. \tag{24b}$$

When resolved in the inertial basis, the corresponding forces are

$$\underline{N}_{\rm et} = 1/2\ c_{\rm et}\kappa_1^{*2}(\underline{\underline{R}}\,\underline{\underline{R}}_0)\bar{\imath}_1, \tag{25a}$$

$$\underline{M}_{\rm et} = c_{\rm et}\kappa_1^{*}\epsilon_1^{*}(\underline{\underline{R}}\,\underline{\underline{R}}_0)\bar{\imath}_1. \tag{25b}$$

The effect of extension-twist coupling is now easily taken into account. In eqs. (20), the force and moment arrays are replaced by $\underline{N} + \underline{N}_{\rm et}$ and $\underline{M} + \underline{M}_{\rm et}$, respectively.

2 Inertial forces

The inertial forces acting in the beam are obtained from the governing equations of motion, eqs. (20),

$$\underline{\mathcal{F}}^I = \dot{\underline{\mathcal{P}}} + \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \widetilde{\dot{\underline{u}}} & \underline{\underline{0}} \end{bmatrix} \underline{\mathcal{P}}, \tag{26}$$

where $\underline{\mathcal{P}}$ is the momentum array resolved in the inertial system. This momentum array is defined in eqs. (18) and can be expressed as

$$\underline{\mathcal{P}} = \underline{\underline{\mathcal{M}}}\,\underline{\mathcal{V}}, \tag{27}$$

where the sectional mass matrix resolved in the inertial system is

$$\underline{\underline{\mathcal{M}}} = (\underline{\underline{\mathcal{R}}}\,\underline{\underline{\mathcal{R}}}_0)\underline{\underline{\mathcal{M}}}^*(\underline{\underline{\mathcal{R}}}\,\underline{\underline{\mathcal{R}}}_0)^T = \begin{bmatrix} m\underline{\underline{I}} & m\widetilde{\eta}^T \\ m\widetilde{\eta} & \underline{\underline{\varrho}} \end{bmatrix}. \tag{28}$$

The location of the sectional center of mass and its moment of inertia tensor, both resolved in the inertial frame, are defined as $\underline{\eta} = (\underline{\underline{R}}\,\underline{\underline{R}}_0)\underline{\eta}^*$, $\underline{\underline{\varrho}} = (\underline{\underline{R}}\,\underline{\underline{R}}_0)\underline{\underline{\varrho}}^*(\underline{\underline{R}}\,\underline{\underline{R}}_0)^T$, respectively.

Expanding eq. (27) now leads to

$$\underline{\mathcal{P}} = \begin{Bmatrix} m\dot{\underline{u}} + m\widetilde{\eta}^T\underline{\omega} \\ m\widetilde{\eta}\dot{\underline{u}} + \underline{\underline{\varrho}}\,\underline{\omega} \end{Bmatrix}. \tag{29}$$

The time derivatives of the location of the sectional mass center and its moment of inertia tensor are $m\dot{\underline{\eta}} = \widetilde{\omega} m\underline{\eta}$, and $\dot{\underline{\underline{\varrho}}} = \widetilde{\omega}\underline{\underline{\varrho}} + \underline{\underline{\varrho}}\widetilde{\omega}^T$, respectively. The time derivative of this momentum array, eq. (27), then becomes

$$\dot{\underline{\mathcal{P}}} = \begin{Bmatrix} m\ddot{\underline{u}} + (\dot{\widetilde{\omega}} + \widetilde{\omega}\widetilde{\omega})m\underline{\eta} \\ m\widetilde{\eta}\ddot{\underline{u}} + \widetilde{\dot{u}}^T\widetilde{\omega}m\underline{\eta} + \widetilde{\omega}\underline{\underline{\varrho}}\,\underline{\omega} + \underline{\underline{\varrho}}\,\dot{\underline{\omega}} \end{Bmatrix}. \tag{30}$$

Finally, the inertial forces, eq. (26), can be written in a compact form as

$$\underline{\mathcal{F}}^I = \begin{Bmatrix} m\ddot{\underline{u}} + (\dot{\widetilde{\omega}} + \widetilde{\omega}\widetilde{\omega})m\underline{\eta} \\ m\widetilde{\eta}\ddot{\underline{u}} + \underline{\underline{\varrho}}\,\dot{\underline{\omega}} + \widetilde{\omega}\underline{\underline{\varrho}}\,\underline{\omega} \end{Bmatrix}. \tag{31}$$

2.1 Linearization of inertial forces

The expression for the inertial forces given above is nonlinear, and the solution process will require linearization of these forces. First, it will be necessary to compute increments of the sectional mass center location and of the sectional moment of inertia tensor, which are found to be $m\Delta\underline{\eta} = m\widetilde{\eta}^T\underline{\Delta\psi}$ and $(\Delta\underline{\underline{\varrho}})\underline{b} = (\underline{\underline{\varrho}}\widetilde{b} - \widetilde{\underline{\underline{\varrho}}\underline{b}})\underline{\Delta\psi}$, respectively, where $\underline{b}$ is an arbitrary vector. Linearization of the inertial forces then yields

$$\Delta\underline{\mathcal{F}}^I = \underline{\underline{\mathcal{K}}}^I \begin{Bmatrix} \Delta\underline{u} \\ \underline{\Delta\psi} \end{Bmatrix} + \underline{\underline{\mathcal{G}}}^I \begin{Bmatrix} \Delta\dot{\underline{u}} \\ \Delta\underline{\omega} \end{Bmatrix} + \underline{\underline{\mathcal{M}}}^I \begin{Bmatrix} \Delta\ddot{\underline{u}} \\ \Delta\dot{\underline{\omega}} \end{Bmatrix}, \tag{32}$$

where $\underline{\underline{\mathcal{K}}}^I$, $\underline{\underline{\mathcal{G}}}^I$, and $\underline{\underline{\mathcal{M}}}^I$ are the stiffness, gyroscopic, and mass matrices associated with the inertial forces, respectively. Simple algebra yields

$$\underline{\underline{\mathcal{K}}}^I = \begin{bmatrix} \underline{\underline{0}} & (\dot{\widetilde{\omega}} + \widetilde{\omega}\widetilde{\omega})m\widetilde{\eta}^T \\ \underline{\underline{0}} & \widetilde{\ddot{u}}m\widetilde{\eta} + (\underline{\underline{\varrho}}\,\widetilde{\dot{\omega}} - \widetilde{\underline{\underline{\varrho}}\dot{\omega}}) + \widetilde{\omega}(\underline{\underline{\varrho}}\,\widetilde{\omega} - \widetilde{\underline{\underline{\varrho}}\omega}) \end{bmatrix}, \tag{33a}$$

$$\underline{\underline{\mathcal{G}}}^I = \begin{bmatrix} \underline{\underline{0}} & \widetilde{\widetilde{\omega}m\eta}^T + \widetilde{\omega}m\widetilde{\eta}^T \\ \underline{\underline{0}} & \widetilde{\omega}\underline{\underline{\varrho}} - \widetilde{\underline{\underline{\varrho}}\omega} \end{bmatrix}, \tag{33b}$$

$$\underline{\underline{\mathcal{M}}}^I = \begin{bmatrix} m\underline{\underline{I}} & m\widetilde{\eta}^T \\ m\widetilde{\eta} & \underline{\underline{\varrho}} \end{bmatrix}. \tag{33c}$$

Here again, these matrices are expressed in terms of physical quantities, the angular velocity and acceleration.

2.2 Inertial forces summary

In summary, the inertial forces given by eq. (31) are

$$\underline{\mathcal{F}}^I = \begin{Bmatrix} m\underline{\ddot{u}} + \dot{\widetilde{\omega}} m\underline{\eta} + \widetilde{\omega}\underline{\beta} \\ m\widetilde{\eta}\underline{\ddot{u}} + \underline{\nu} + \widetilde{\omega}\underline{\gamma} \end{Bmatrix}. \tag{34}$$

The stiffness, gyroscopic, and mass matrices given by eq. (33a), (33b), and (33c) become

$$\underline{\underline{\mathcal{K}}}^I = \begin{bmatrix} \underline{\underline{0}} & \dot{\widetilde{\omega}} m \widetilde{\eta}^T + \widetilde{\omega}\underline{\underline{\mu}} \\ \underline{\underline{0}} & \ddot{\widetilde{u}} m\widetilde{\eta} + \underline{\underline{\varrho}}\,\dot{\widetilde{\omega}} - \widetilde{\nu} + \underline{\underline{\varepsilon}}\widetilde{\omega} - \widetilde{\omega}\widetilde{\gamma} \end{bmatrix}, \tag{35a}$$

$$\underline{\underline{\mathcal{G}}}^I = \begin{bmatrix} \underline{\underline{0}} & \widetilde{\beta}^T + \underline{\underline{\mu}} \\ \underline{\underline{0}} & \underline{\underline{\varepsilon}} - \widetilde{\gamma} \end{bmatrix}, \tag{35b}$$

$$\underline{\underline{\mathcal{M}}}^I = \begin{bmatrix} m\underline{\underline{I}} & m\widetilde{\eta}^T \\ m\widetilde{\eta} & \underline{\underline{\varrho}} \end{bmatrix}. \tag{35c}$$

The following notations were introduced

$$\underline{\beta} = \widetilde{\omega} m \underline{\eta}, \quad \underline{\gamma} = \underline{\underline{\varrho}}\,\underline{\omega}, \quad \underline{\nu} = \underline{\underline{\varrho}}\,\underline{\dot{\omega}}, \quad \underline{\underline{\varepsilon}} = \widetilde{\omega}\,\underline{\underline{\varrho}}, \quad \underline{\underline{\mu}} = \widetilde{\omega} m \widetilde{\eta}^T. \tag{36}$$

3 Elastic forces

The elastic forces acting in the beam element are defined in eqs. (20) and will be treated in two separate components, denoted $\underline{\mathcal{F}}^C$ and $\underline{\mathcal{F}}^D$, defined as

$$\underline{\mathcal{F}}^C = \underline{f} = \begin{Bmatrix} \underline{N} \\ \underline{M} \end{Bmatrix}, \text{ and } \underline{\mathcal{F}}^D = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \widetilde{E}_1^T & \underline{\underline{0}} \end{bmatrix} \underline{f} = \begin{Bmatrix} \underline{0} \\ \widetilde{E}_1^T \underline{N} \end{Bmatrix}, \tag{37}$$

respectively, where $\underline{E}_1 = \underline{u}_0' + \underline{u}'$. The components of the beam's sectional force and moment vectors resolved in the inertial basis are denoted $\underline{N}$ and $\underline{M}$, respectively.

The sectional strains and curvatures are recast in the following compact notation

$$\underline{e} = \begin{Bmatrix} \underline{E}_1 - (\underline{\underline{R}}\,\underline{\underline{R}}_0)\bar{\imath}_1 \\ \underline{k} \end{Bmatrix}, \tag{38}$$

where $\underline{k} = \text{axial}(\underline{\underline{R}}'\underline{\underline{R}}^T)$ are the components of the sectional curvature vector resolved in the inertial basis. The corresponding strain components resolved in the material basis are

$$\underline{e}^* = \begin{Bmatrix} (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T \underline{E}_1 - \bar{\imath}_1 \\ (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T \underline{k} \end{Bmatrix}. \tag{39}$$

The elastic forces in the beam are then $\underline{f} = \underline{\underline{\mathcal{C}}}\,\underline{e}$, where $\underline{\underline{\mathcal{C}}} = (\underline{\underline{R}}\,\underline{\underline{R}}_0)\underline{\underline{\mathcal{C}}}^*(\underline{\underline{R}}\,\underline{\underline{R}}_0)^T$ is the sectional stiffness matrix resolved in the inertial basis, and the corresponding stiffness matrix resolved in the material basis is denoted $\underline{\underline{\mathcal{C}}}^*$.

3.1 Linearization of elastic forces

The expressions for the elastic forces given above are nonlinear, and the finite element process will require a linearization of these forces. First, increments in the curvature vector are easily found as $\Delta\underline{k} = \underline{\Delta\psi}' - \widetilde{k}\underline{\Delta\psi}$. Increments in the strain components, eq. (38), are now easily evaluated to find

$$\Delta\underline{e} = \begin{Bmatrix} \Delta\underline{u}' + (\widetilde{\underline{\underline{R}}\,\underline{\underline{R}}_0)\bar{\imath}_1}\underline{\Delta\psi} \\ \underline{\Delta\psi}' - \widetilde{k}\underline{\Delta\psi} \end{Bmatrix}.$$

This leads to the following expression for increments in the elastic forces

$$\Delta\underline{f} = \begin{Bmatrix} \widetilde{N}^T\underline{\Delta\psi} \\ \widetilde{M}^T\underline{\Delta\psi} \end{Bmatrix} + \underline{\underline{\mathcal{C}}} \begin{Bmatrix} \widetilde{E}_1\underline{\Delta\psi} + \Delta\underline{u}' \\ \underline{\Delta\psi}' \end{Bmatrix}.$$

Taking variations of eq. (37) yields the following expression for increments in the elastic forces

$$\Delta\underline{\mathcal{F}}^C = \underline{\underline{\mathcal{C}}} \begin{Bmatrix} \Delta\underline{u}' \\ \underline{\Delta\psi}' \end{Bmatrix} + \underline{\underline{\mathcal{O}}} \begin{Bmatrix} \Delta\underline{u} \\ \underline{\Delta\psi} \end{Bmatrix}, \quad \Delta\underline{\mathcal{F}}^D = \underline{\underline{\mathcal{P}}} \begin{Bmatrix} \Delta\underline{u}' \\ \underline{\Delta\psi}' \end{Bmatrix} + \underline{\underline{\mathcal{Q}}} \begin{Bmatrix} \Delta\underline{u} \\ \underline{\Delta\psi} \end{Bmatrix}, \tag{40}$$

where the stiffness matrices are

$$\underline{\underline{\mathcal{O}}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{C}}_{11}\widetilde{E}_1 - \widetilde{N} \\ \underline{\underline{0}} & \underline{\underline{C}}_{21}\widetilde{E}_1 - \widetilde{M} \end{bmatrix}, \quad \underline{\underline{\mathcal{P}}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \widetilde{N} + (\underline{\underline{C}}_{11}\widetilde{E}_1)^T & (\underline{\underline{C}}_{21}\widetilde{E}_1)^T \end{bmatrix}, \quad \underline{\underline{\mathcal{Q}}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \widetilde{E}_1^T \underline{\underline{O}}_{12} \end{bmatrix}. \tag{41}$$

The following partitions were introduced

$$\underline{\underline{\mathcal{C}}} = \begin{bmatrix} \underline{\underline{C}}_{11} & \underline{\underline{C}}_{12} \\ \underline{\underline{C}}_{21} & \underline{\underline{C}}_{22} \end{bmatrix}, \quad \underline{\underline{\mathcal{O}}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{O}}_{12} \\ \underline{\underline{0}} & \underline{\underline{O}}_{22} \end{bmatrix}. \tag{42}$$

3.2 Elastic forces summary

In summary, the elastic forces given by eq. (37) are

$$\underline{\mathcal{F}}^C = \underline{\underline{\mathcal{C}}}\,\underline{e} = \begin{Bmatrix} \underline{N} \\ \underline{M} \end{Bmatrix}, \quad \underline{\mathcal{F}}^D = \begin{Bmatrix} \underline{0} \\ \widetilde{E}_1^T \underline{N} \end{Bmatrix}. \tag{43}$$

The stiffness matrices given by eqs. (41) become

$$\underline{\underline{\mathcal{O}}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{\varepsilon}} - \widetilde{N} \\ \underline{\underline{0}} & \underline{\underline{\mu}} - \widetilde{M} \end{bmatrix}, \tag{44a}$$

$$\underline{\underline{\mathcal{P}}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \widetilde{N} + \underline{\underline{\varepsilon}}^T & \underline{\underline{\mu}}^T \end{bmatrix}, \tag{44b}$$

$$\underline{\underline{\mathcal{Q}}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \widetilde{E}_1^T \underline{\underline{O}}_{12} \end{bmatrix}. \tag{44c}$$

The following notations were defined

$$\underline{\underline{\varepsilon}} = \underline{\underline{C}}_{11}\widetilde{E}_1, \quad \underline{\underline{\mu}} = \underline{\underline{C}}_{21}\widetilde{E}_1. \tag{45}$$

3.3 Effect of extension-twist coupling

As discussed in section 1.4, the effect of extension-twist coupling is that the force and moment arrays are replaced by $\underline{N}+\underline{N}_{\rm et}$ and $\underline{M}+\underline{M}_{\rm et}$, respectively. Introducing this substitution in the elastic forces and stiffness matrices presented in the previous section will account for the effect of extension-twist coupling.

The extension-twist forces and moments should also be linearized to yield the consistent tangent stiffness matrices. This leads to the following updated stiffness matrix resolved in the material basis

$$\underline{\underline{\mathcal{C}}}^* + c_{\rm et} \begin{bmatrix} \underline{\underline{0}} & \kappa_1^* \bar{\imath}_1 \bar{\imath}_1^T \\ \kappa_1^* \bar{\imath}_1 \bar{\imath}_1^T & \epsilon_1^* \bar{\imath}_1 \bar{\imath}_1^T \end{bmatrix}, \tag{46}$$

where the second term represents the effect of the extension-twist coupling.

4 Dissipative forces

The beam model discussed in the previous section is a conservative model, because the elastic forces are proportional the strain measures. It is often desirable to also introduce dissipative forces in the beam model. The dissipative forces in the material frame, denoted $\underline{f}_d^*$, will be written as

$$\underline{f}_d^* = \begin{Bmatrix} \underline{N}_d^* \\ \underline{M}_d^* \end{Bmatrix} = \mu \underline{\underline{\mathcal{C}}}^* \underline{\dot{e}}^*, \tag{47}$$

where μ is the damping coefficient of units 1/s, and $\underline{\dot{e}}^*$ the time rate of change of the strains resolved in the material frame. Since the dissipative mechanisms in the beam are not well understood, it is postulated that the damping matrix is proportional to the stiffness matrix. The time rate of change of the sectional strains in the material frame are readily obtained from eq. (39) as

$$\underline{\dot{e}}^* = \begin{Bmatrix} (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T(\underline{\dot{u}}' + \widetilde{E}_1\underline{\omega}) \\ (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T(\underline{\dot{k}} + \widetilde{k}\underline{\omega}) \end{Bmatrix} = \begin{Bmatrix} (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T(\underline{\dot{u}}' + \widetilde{E}_1\underline{\omega}) \\ (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T\underline{\omega}' \end{Bmatrix}, \tag{48}$$

where the second equality results from the following identity, $\underline{\dot{k}} - \underline{\omega}' = \widetilde{\omega}\underline{k}$.

The dissipative forces resolved in the inertial frame now become

$$\underline{f}_d = \mu \underline{\underline{\mathcal{C}}}\, \underline{\dot{e}}, \tag{49}$$

where $\underline{f}_d^T = \{\underline{N}_d^T, \underline{M}_d^T\}$, and $\underline{N}_d = (\underline{\underline{R}}\,\underline{\underline{R}}_0)\underline{N}_d^*$ and $\underline{M}_d = (\underline{\underline{R}}\,\underline{\underline{R}}_0)\underline{M}_d^*$ are the are the sectional dissipative force and moment vector components in the inertial frame, respectively, $\underline{\underline{\mathcal{C}}} = (\underline{\underline{\mathcal{R}}}\,\underline{\underline{\mathcal{R}}}_0)\underline{\underline{\mathcal{C}}}^*(\underline{\underline{\mathcal{R}}}\,\underline{\underline{\mathcal{R}}}_0)^T$ is the sectional stiffness matrix resolved in the inertial basis, and $\underline{\dot{e}}$ follows from eq. (48) as

$$\underline{\dot{e}} = \begin{Bmatrix} \underline{\dot{u}}' + \widetilde{E}_1 \underline{\omega} \\ \underline{\omega}' \end{Bmatrix}. \tag{50}$$

The dissipative forces will be treated in two separate components, denoted $\underline{\mathcal{F}}_d^C$ and $\underline{\mathcal{F}}_d^D$,

$$\underline{\mathcal{F}}_d^C = \underline{f}_d = \begin{Bmatrix} \underline{N}_d \\ \underline{M}_d \end{Bmatrix}, \quad \underline{\mathcal{F}}_d^D = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \widetilde{E}_1^T & \underline{\underline{0}} \end{bmatrix} \underline{f}_d = \begin{Bmatrix} \underline{0} \\ \widetilde{E}_1^T \underline{N}_d \end{Bmatrix}. \tag{51}$$

4.1 Linearization of dissipative forces

Because the expression for the dissipative forces is nonlinear, the solution process will require a linearization. Increments in the strain array are evaluated to find

$$\Delta \underline{\dot{e}} = \begin{Bmatrix} \Delta \underline{\dot{u}}' + \widetilde{E}_1(\Delta \underline{\dot{\psi}} - \widetilde{\omega}\Delta\underline{\psi}) + \widetilde{\omega}^T \Delta \underline{u}' \\ \Delta \underline{\dot{\psi}}' - \widetilde{\omega}\Delta\underline{\psi}' - \widetilde{\omega}'\Delta\underline{\psi} \end{Bmatrix}. \tag{52}$$

Next, increments in the dissipative forces are computed

$$\Delta \underline{f}_d = \begin{Bmatrix} \widetilde{N}_d^T \Delta\underline{\psi} \\ \widetilde{M}_d^T \Delta\underline{\psi} \end{Bmatrix} + \mu \underline{\underline{\mathcal{C}}} \left\{ \begin{bmatrix} \underline{\underline{0}} & \dot{\widetilde{u}}' - \widetilde{\omega}\widetilde{E}_1 \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \begin{Bmatrix} \Delta\underline{u} \\ \Delta\underline{\psi} \end{Bmatrix} + \begin{bmatrix} \widetilde{\omega}^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \widetilde{\omega}^T \end{bmatrix} \begin{Bmatrix} \Delta\underline{u}' \\ \Delta\underline{\psi}' \end{Bmatrix} + \begin{bmatrix} \underline{\underline{0}} & \widetilde{E}_1 \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \begin{Bmatrix} \Delta\underline{\dot{u}} \\ \Delta\underline{\dot{\psi}} \end{Bmatrix} + \begin{Bmatrix} \Delta\underline{\dot{u}}' \\ \Delta\underline{\dot{\psi}}' \end{Bmatrix} \right\}. \tag{53}$$

Taking variations of eq. (51) yields the following expression for increments in the dissipative forces

$$\Delta \underline{\mathcal{F}}_d^C = \underline{\underline{\mathcal{S}}}_d \begin{Bmatrix} \Delta\underline{u}' \\ \Delta\underline{\psi}' \end{Bmatrix} + \underline{\underline{\mathcal{O}}}_d \begin{Bmatrix} \Delta\underline{u} \\ \Delta\underline{\psi} \end{Bmatrix} + \underline{\underline{\mathcal{G}}}_d \begin{Bmatrix} \Delta\underline{\dot{u}} \\ \Delta\underline{\dot{\psi}} \end{Bmatrix} + \mu\underline{\underline{\mathcal{C}}}\Delta \begin{Bmatrix} \Delta\underline{\dot{u}}' \\ \Delta\underline{\dot{\psi}}' \end{Bmatrix}, \tag{54a}$$

$$\Delta \underline{\mathcal{F}}_d^D = \underline{\underline{\mathcal{P}}}_d \begin{Bmatrix} \Delta\underline{u}' \\ \Delta\underline{\psi}' \end{Bmatrix} + \underline{\underline{\mathcal{Q}}}_d \begin{Bmatrix} \Delta\underline{u} \\ \Delta\underline{\psi} \end{Bmatrix} + \underline{\underline{\mathcal{X}}}_d \begin{Bmatrix} \Delta\underline{\dot{u}} \\ \Delta\underline{\dot{\psi}} \end{Bmatrix} + \underline{\underline{\mathcal{Y}}}_d\Delta \begin{Bmatrix} \Delta\underline{\dot{u}}' \\ \Delta\underline{\dot{\psi}}' \end{Bmatrix}. \tag{54b}$$

In summary, the dissipative forces can be written in the following form

$$\underline{\mathcal{F}}_d^C = \mu\underline{\underline{\mathcal{C}}}\,\underline{\dot{e}} = \begin{Bmatrix} \underline{N}_d \\ \underline{M}_d \end{Bmatrix}, \quad \underline{\mathcal{F}}_d^D = \begin{Bmatrix} \underline{0} \\ \widetilde{E}_1^T \underline{N}_d \end{Bmatrix}, \tag{55}$$

where the gyroscopic and stiffness matrices are

$$\underline{\underline{\mathcal{S}}}_d = \mu\mathcal{C} \begin{bmatrix} \widetilde{\omega}^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \widetilde{\omega}^T \end{bmatrix}, \quad \underline{\underline{\mathcal{O}}}_d = \begin{bmatrix} \underline{\underline{0}} & \mu\underline{\underline{C}}_{11}\underline{\underline{\alpha}} - \widetilde{N}_d \\ \underline{\underline{0}} & \mu\underline{\underline{C}}_{21}\underline{\underline{\alpha}} - \widetilde{M}_d \end{bmatrix}, \quad \underline{\underline{\mathcal{G}}}_d = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{\beta}}_{11}^T \\ \underline{\underline{0}} & \underline{\underline{\beta}}_{12}^T \end{bmatrix}, \tag{56}$$

and

$$\underline{\underline{\mathcal{P}}}_d = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \widetilde{N}_d + \underline{\underline{\beta}}_{11}\widetilde{\omega}^T & \underline{\underline{\beta}}_{12}\widetilde{\omega}^T \end{bmatrix}, \quad \underline{\underline{\mathcal{Q}}}_d = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \widetilde{E}_1^T \underline{\underline{O}}_{12} \end{bmatrix}, \quad \underline{\underline{\mathcal{X}}}_d = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \widetilde{E}_1^T \underline{\underline{G}}_{12} \end{bmatrix}, \quad \underline{\underline{\mathcal{Y}}}_d = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{\beta}}_{11} & \underline{\underline{\beta}}_{12} \end{bmatrix}. \tag{57}$$

The following notation was introduced

$$\underline{\underline{\alpha}} = \dot{\widetilde{u}}' - \widetilde{\omega}\widetilde{E}_1, \quad \underline{\underline{\beta}}_{11} = \widetilde{E}_1^T \mu\underline{\underline{C}}_{11}, \quad \underline{\underline{\beta}}_{12} = \widetilde{E}_1^T \mu\underline{\underline{C}}_{12}. \tag{58}$$

5 Gravity forces

For many applications, the gravity forces associated with the beam's distributed mass must be taken into account. The potential of these gravity forces is $V = m\underline{g}^T(\underline{u}_0 + \underline{u} + \underline{\eta})$, where vector $\underline{\eta}$ defines the location of the sectional mass center.

A variation of this potential is easily found to be $\delta V = \underline{g}^T(m\delta\underline{u} + m\widetilde{\eta}^T\underline{\delta\psi})$ and the gravity forces acting on the cross-section are readily found as

$$\underline{\mathcal{F}}^G = \begin{Bmatrix} m\underline{g} \\ m\widetilde{\eta}\underline{g} \end{Bmatrix}. \tag{59}$$

6 Rigid rotation forces

It is sometime desirable to perform a static analysis, *i.e.* an analysis that neglects inertial forces. This can be easily achieved setting all time derivatives to zero in the equations of motion derived in the previous paragraphs, and solve the resulting nonlinear algebraic system. Furthermore, it is also possible to perform an eigenvalue analysis to determine the natural frequencies of the structure. In that case, the mass, gyroscopic, and stiffness matrices of the system must be evaluated.

6.1 Rotating structure

An interesting case to consider is that where one of more subcomponents of the structure to be analyzed are rotating at a constant angular velocity with respect to other subcomponents. While such problem is not strictly speaking a static problem because the rotating subcomponents feature non-vanishing linear and angular velocities, it can be formulated as a static problem where the centrifugal forces generated by the rigid body rotation are applied on the otherwise static structure.

Figure 2 shows a non-rotating structure, a simple cantilevered beam in this case, with a rotating subcomponent, a tip rotating beam. Frame $\mathcal{F}^I = [\mathbf{O}, \mathcal{I} = (\bar\imath_1, \bar\imath_2, \bar\imath_3)]$ is the inertial frame, frame $\mathcal{F}_0^A = [\mathbf{A}, \mathcal{B}_0^A = (\bar b_{10}^A, \bar b_{20}^A, \bar b_{30}^A)]$ is attached to the non-rotating structure at point **A** in its reference configuration, and frame $\mathcal{F}^A = [\mathbf{A}, \mathcal{B}^A = (\bar b_1^A, \bar b_2^A, \bar b_3^A)]$ is attached to the non-rotating structure at point **A** in the deformed configuration. Frame $\mathcal{F}^R = [\mathbf{A}, \mathcal{B}^R = (\bar b_1^R, \bar b_2^R, \bar b_3^R)]$ rotates at a constant angular velocity, $\underline{\omega}_r$, with respect to frame $\mathcal{F}^A$.

Let $\underline{\underline{R}}_0^A$ and $\underline{\underline{R}}^A$ be the components of the rotation tensors that bring triad $\mathcal{I}$ to $\mathcal{B}_0^A$ and triad $\mathcal{B}_0^A$ to $\mathcal{B}^A$, respectively, both resolved in the inertial basis, $\mathcal{I}$. Furthermore, let $\underline{\underline{R}}_r^*$ be the components of the rotation tensor describing the finite rotation from triad $\mathcal{B}^A$ to $\mathcal{B}^R$, resolved in basis $\mathcal{B}^A$. With these definition, it follows that

$$\bar b_1^R = (\underline{\underline{R}}^A \underline{\underline{R}}_0^A)\underline{\underline{R}}_r^* \bar\imath_1. \tag{60}$$

The time derivative of this unit vector is readily computed as

$$\dot{\bar b}_1^R = \widetilde{\omega}_r \bar b_1^R, \tag{61}$$

where $\widetilde{\omega}_r^* = \dot{\underline{\underline{R}}}_r^* \underline{\underline{R}}_r^{*T}$ are the components of the relative angular velocity of the rotating subcomponent with respect to its non-rotating counterpart resolved in basis $\mathcal{B}^A$, and $\underline{\omega}_r = (\underline{\underline{R}}^A \underline{\underline{R}}_0^A)\underline{\omega}_r^*$ the components of the same vector resolved in the inertial basis. It is assumed here that the rotating subcomponent rotates at a given, constant angular velocity with respect to the non-rotating subcomponent, *i.e.* $\underline{\omega}_r^*$ is a constant vector in time. Note that for a static problem, components $\underline{\omega}_r$ are also constant, although not known, because the motion of frame $\mathcal{F}^A$ is itself unknown.

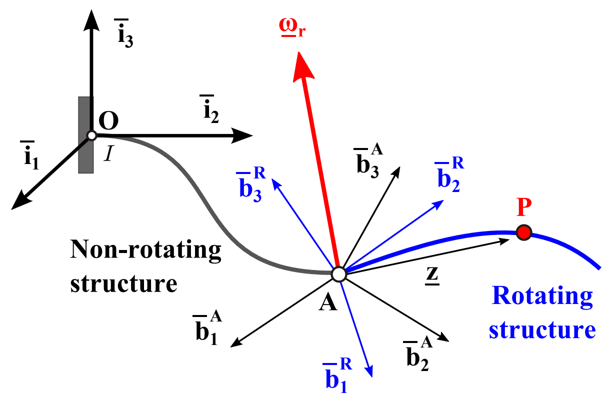

Figure 2: A structure featuring a subcomponent rotating at a constant angular velocity $\underline{\omega}_r$.

6.2 Rigid body rotation forces

Consider now an arbitrary point **P** of the rotating structure and its position vector, $\underline{z}$, with respect to point **A**, as shown in fig. 2. Since the motion of the rotating structure is a rigid body rotation, the inertial velocity of point **P** is simply $\underline{\dot u} = \widetilde{\omega}_r \underline{z}$. The components of this vector in the rotating frame, $\underline{\dot u}^* = \underline{\underline{R}}_r^{*T}(\underline{\underline{R}}^A \underline{\underline{R}}_0^A)^T \underline{\dot u}$, are constant because the motion of the rotating structure is at a constant angular velocity, $\underline{\omega}_r^*$. This implies $R_r^{*T}\widetilde{\omega}_r^*(\underline{\underline{R}}^A \underline{\underline{R}}_0^A)^T \underline{\dot u} + \underline{\underline{R}}_r^{*T}(\underline{\underline{R}}^A \underline{\underline{R}}_0^A)^T \underline{\ddot u} = 0$, and hence $\underline{\ddot u} = \widetilde{\omega}_r \underline{\dot u}$. In summary, the velocity and acceleration fields of the rotating structure are

$$\begin{Bmatrix} \underline{\dot u} \\ \underline{\omega} \end{Bmatrix} = \begin{Bmatrix} \widetilde{\omega}_r \underline{z} \\ \underline{\omega}_r \end{Bmatrix}, \quad \text{and} \quad \begin{Bmatrix} \underline{\ddot u} \\ \underline{\dot\omega} \end{Bmatrix} = \begin{Bmatrix} \widetilde{\omega}_r \widetilde{\omega}_r \underline{z} \\ 0 \end{Bmatrix}, \tag{62}$$

respectively.

The forces associated with the rigid body rotation are readily found by inserting these fields into eq. (31) to find the centrifugal forces applied to the rotating structure as

$$\underline{\mathcal{F}}_r = \begin{Bmatrix} m\widetilde{\omega}_r\widetilde{\omega}_r\underline{z} + \widetilde{\omega}_r\widetilde{\omega}_r m\underline{\eta} \\ m\widetilde{\eta}\widetilde{\omega}_r\widetilde{\omega}_r\underline{z} + \widetilde{\omega}_r\underline{\underline{\varrho}}\underline{\omega}_r \end{Bmatrix}. \tag{63}$$

Note that these are the "static forces," *i.e.* time independent forces, applied to the structure due to its rigid body rotation. In a static analysis, the steady centrifugal forces given by eq. (63) are applied to the non-rotating structure. In other words, for the static analysis, the structure is not rotating, but subjected to the steady centrifugal loading that would appear *if the structure were actually rotating*.

6.3 Linearization of the rigid body rotation forces

Although the rigid body rotation considered here is characterized by an angular velocity vector of constant magnitude, the orientation of this angular velocity vector is unknown as the structure is allowed to deform statically, *i.e.*, the rotation tensor at point **A** is unknown. Similarly, the relative position vector $\underline{z}$ shown in fig. 2 is not known because it is a function of the displacements at points **P** and **A**. These facts are expressed by the following relationships,

$$\Delta\underline{\omega}_r = \widetilde{\omega}_r^T \Delta\underline{\psi}_A, \tag{64a}$$

$$\Delta\underline{z} = \Delta\underline{u} - \Delta\underline{u}_A, \tag{64b}$$

where $\Delta\underline{u}_A$ and $\Delta\underline{\psi}_A$ are small increments in the position and rotation at point **A**.

To solve the nonlinear static problem, a linearization process similar to that described in section 2.1 must be performed. The linearization of the inertial forces associated with the rigid body motion can be cast as

$$\Delta\underline{\mathcal{F}}_r = \underline{\underline{\bar{\mathcal{K}}}}_r \begin{Bmatrix} \Delta\underline{u} \\ \Delta\underline{\psi} \end{Bmatrix} + \underline{\underline{\bar{\mathcal{G}}}}_r \left(\begin{Bmatrix} \Delta\underline{\dot{u}} \\ \Delta\underline{\omega} \end{Bmatrix} + \begin{Bmatrix} \Delta(\widetilde{\omega}_r\underline{z}) \\ \Delta\underline{\omega}_r \end{Bmatrix} \right) + \underline{\underline{\bar{\mathcal{M}}}}_r \left(\begin{Bmatrix} \Delta\underline{\ddot{u}} \\ \Delta\underline{\dot{\omega}} \end{Bmatrix} + \begin{Bmatrix} \Delta(\widetilde{\omega}_r\widetilde{\omega}_r\underline{z}) \\ \underline{0} \end{Bmatrix} \right), \tag{65}$$

where $\underline{\underline{\bar{\mathcal{K}}}}_r$, $\underline{\underline{\bar{\mathcal{G}}}}_r$, and $\underline{\underline{\bar{\mathcal{M}}}}_r$ are the stiffness, gyroscopic, and mass matrices associated with the rigid rotation forces, respectively, obtained by introducing the velocity and acceleration fields associated with the rigid body rotation, eqs. (62), into the corresponding matrices defined in eqs. (33a), (33b), and (33c) to find

$$\underline{\underline{\bar{\mathcal{K}}}}_r = \begin{bmatrix} \underline{\underline{0}} & \widetilde{\omega}_r\widetilde{\omega}_r m\widetilde{\eta}^T \\ \underline{\underline{0}} & \widetilde{\widetilde{\omega}_r\widetilde{\omega}_r\underline{z}}\, m\widetilde{\eta} + \widetilde{\omega}_r(\underline{\underline{\varrho}}\,\widetilde{\omega}_r - \widetilde{\underline{\underline{\varrho}}\underline{\omega}_r}) \end{bmatrix}, \tag{66a}$$

$$\underline{\underline{\bar{\mathcal{G}}}}_r = \begin{bmatrix} \underline{\underline{0}} & \widetilde{\widetilde{\omega}_r m\underline{\eta}}^T + \widetilde{\omega}_r m\widetilde{\eta}^T \\ \underline{\underline{0}} & \widetilde{\omega}_r\underline{\underline{\varrho}} - \widetilde{\underline{\underline{\varrho}}\underline{\omega}_r} \end{bmatrix}, \tag{66b}$$

$$\underline{\underline{\bar{\mathcal{M}}}}_r = \begin{bmatrix} m\underline{\underline{I}} & m\widetilde{\eta}^T \\ m\widetilde{\eta} & \underline{\underline{\varrho}} \end{bmatrix}. \tag{66c}$$

The second terms appearing in the two parentheses of eq. (65) correspond to perturbations in the rigid body rotation. With the help of eqs. (64), these perturbations are expressed as

$$\begin{Bmatrix} \Delta(\widetilde{\omega}_r\underline{z}) \\ \Delta\underline{\omega}_r \end{Bmatrix} = \begin{bmatrix} \widetilde{\omega}_r & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \begin{Bmatrix} \Delta\underline{u} \\ \Delta\underline{\psi} \end{Bmatrix} + \begin{bmatrix} \widetilde{\omega}_r^T & \widetilde{z}^T\widetilde{\omega}_r^T \\ \underline{\underline{0}} & \widetilde{\omega}_r^T \end{bmatrix} \begin{Bmatrix} \Delta\underline{u}_A \\ \Delta\underline{\psi}_A \end{Bmatrix}, \tag{67a}$$

$$\begin{Bmatrix} \Delta(\widetilde{\omega}_r\widetilde{\omega}_r\underline{z}) \\ \underline{0} \end{Bmatrix} = \begin{bmatrix} \widetilde{\omega}_r\widetilde{\omega}_r & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \begin{Bmatrix} \Delta\underline{u} \\ \Delta\underline{\psi} \end{Bmatrix} + \begin{bmatrix} \widetilde{\omega}_r\widetilde{\omega}_r^T & (\widetilde{z}\widetilde{\omega}_r - 2\widetilde{\omega}_r\widetilde{z})\widetilde{\omega}_r^T \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \begin{Bmatrix} \Delta\underline{u}_A \\ \Delta\underline{\psi}_A \end{Bmatrix} \tag{67b}$$

Introducing these results into eq. (65) then yields the desired linearization of the rigid body rotation forces

$$\Delta\underline{\mathcal{F}}_r = \underline{\underline{\mathcal{M}}}_r \begin{Bmatrix} \Delta\underline{\ddot{u}} \\ \Delta\underline{\dot{\omega}} \end{Bmatrix} + \underline{\underline{\mathcal{G}}}_r \begin{Bmatrix} \Delta\underline{\dot{u}} \\ \Delta\underline{\omega} \end{Bmatrix} + \underline{\underline{\mathcal{K}}}_r \begin{Bmatrix} \Delta\underline{u} \\ \Delta\underline{\psi} \end{Bmatrix} + \underline{\underline{\mathcal{K}}}_A \begin{Bmatrix} \Delta\underline{u}_A \\ \Delta\underline{\psi}_A \end{Bmatrix}, \tag{68}$$

where the following matrices were defined

$$\underline{\underline{\mathcal{K}}}_r = \underline{\underline{\bar{\mathcal{K}}}}_r + \underline{\underline{\bar{\mathcal{M}}}}_r \begin{bmatrix} \widetilde{\omega}_r\widetilde{\omega}_r & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} + \underline{\underline{\bar{\mathcal{G}}}}_r \begin{bmatrix} \widetilde{\omega}_r & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} = \underline{\underline{\bar{\mathcal{K}}}}_r + \begin{bmatrix} m\widetilde{\omega}_r\widetilde{\omega}_r & \underline{\underline{0}} \\ m\widetilde{\eta}\widetilde{\omega}_r\widetilde{\omega}_r & \underline{\underline{0}} \end{bmatrix}, \tag{69a}$$

$$\underline{\underline{\mathcal{G}}}_r = \underline{\underline{\bar{\mathcal{G}}}}_r, \tag{69b}$$

$$\underline{\underline{\mathcal{M}}}_r = \underline{\underline{\bar{\mathcal{M}}}}_r, \tag{69c}$$

$$\underline{\underline{\mathcal{K}}}_A = \underline{\underline{\bar{\mathcal{M}}}}_r \begin{bmatrix} \widetilde{\omega}_r\widetilde{\omega}_r^T & (\widetilde{z}\widetilde{\omega}_r - 2\widetilde{\omega}_r\widetilde{z})\widetilde{\omega}_r^T \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} + \underline{\underline{\bar{\mathcal{G}}}}_r \begin{bmatrix} \widetilde{\omega}_r^T & \widetilde{z}^T\widetilde{\omega}_r^T \\ \underline{\underline{0}} & \widetilde{\omega}_r^T. \end{bmatrix} \tag{69d}$$

$$\bar{K}_A = \begin{bmatrix} m\widetilde{\omega}_r\widetilde{\omega}_r^T & \left[\underline{\omega}_r^T m(\underline{z}+\underline{\eta}) + \underline{\omega}_r m(\underline{z}+\underline{\eta})^T\right]\widetilde{\omega}_r^T \\ m\widetilde{\eta}\widetilde{\omega}_r\widetilde{\omega}_r^T & \left[m\widetilde{\eta}(\underline{\omega}_r^T\underline{z} + \underline{\omega}_r\underline{z}^T) + \widetilde{\omega}_r\underline{\underline{\varrho}} - \widetilde{\underline{\underline{\varrho}}\underline{\omega}_r}\right]\widetilde{\omega}_r^T \end{bmatrix}. \tag{70}$$

7 Finite element implementation

With the notation defined in eqs. (26) and (37), the equations of motion of curved beams can be recast in the following compact form, $\underline{\mathcal{F}}^I - \underline{\mathcal{F}}^{C\prime} + \underline{\mathcal{F}}^D = \underline{\mathcal{F}}^G + \underline{\mathcal{F}}^{\text{ext}}$, where $\underline{\mathcal{F}}^{\text{ext}}$ are the external forces applied to the beam.

A weighted residual formulation will be used here to enforce these dynamic equilibrium conditions

$$\int_0^\ell \underline{\underline{N}}^T \left(\underline{\mathcal{F}}^I - \underline{\mathcal{F}}^{C\prime} + \underline{\mathcal{F}}^D - \underline{\mathcal{F}}^G - \underline{\mathcal{F}}^{\mathrm{ext}}\right) \mathrm{d}\alpha_1 = 0,$$

where ℓ is the length of the beam element and $\underline{\underline{N}}$ a matrix storing the selected test functions. An integration by parts is performed on the second term of this equation, leading to

$$\int_0^\ell \left(\underline{\underline{N}}^T \underline{\mathcal{F}}^I + \underline{\underline{N}}'^T \underline{\mathcal{F}}^C + \underline{\underline{N}}^T \underline{\mathcal{F}}^D\right) \,\mathrm{d}\alpha_1 = \int_0^\ell \underline{\underline{N}}^T (\underline{\mathcal{F}}^G + \underline{\mathcal{F}}^{\mathrm{ext}}) \,\mathrm{d}x_1.$$

Since this set of algebraic equations is nonlinear, a linearization process is required to solve it. Equation (40) is introduced to find

$$\int_0^\ell \Big[\underline{\underline{N}}^T \Big(\underline{\mathcal{F}}^I + \underline{\underline{\mathcal{K}}}^I \Delta \underline{q} + \underline{\underline{\mathcal{G}}}^I \Delta \underline{v} + \underline{\underline{\mathcal{M}}} \Delta \underline{a} + \underline{\mathcal{F}}^D + \underline{\underline{\mathcal{P}}} \Delta \underline{q}' + \underline{\underline{\mathcal{Q}}} \Delta \underline{q}\Big)$$
$$+ \underline{\underline{N}}'^T \left(\underline{\mathcal{F}}^C + \underline{\underline{\mathcal{S}}} \Delta \underline{q}' + \underline{\underline{\mathcal{O}}} \Delta \underline{q}\right)\Big] \,\mathrm{d}\alpha_1 = \int_0^\ell \underline{\underline{N}}^T (\underline{\mathcal{F}}^G + \underline{\mathcal{F}}^{\mathrm{ext}}) \,\mathrm{d}\alpha_1.$$

Next, the elemental displacement, velocity, and acceleration fields are expressed in terms of their nodal values using the assumed shape functions, $\underline{q}(x_1) = \underline{\underline{N}}\,\hat{\underline{q}}$, $\underline{q}'(x_1) = \underline{\underline{N}}'\hat{\underline{q}}$, $\underline{v}(x_1) = \underline{\underline{N}}\,\hat{\underline{v}}$, $\underline{a}(x_1) = \underline{\underline{N}}\,\hat{\underline{a}}$, where $\hat{\underline{q}}$, $\hat{\underline{v}}$, and $\hat{\underline{a}}$ are the nodal values of the displacements, velocities, and accelerations, respectively. With the help of these interpolations of elemental fields, the weak statement of dynamic equilibrium becomes

$$\hat{\underline{\underline{M}}} \Delta \hat{\underline{a}} + \hat{\underline{\underline{G}}} \Delta \hat{\underline{v}} + \hat{\underline{\underline{K}}} \Delta \hat{\underline{q}} = \hat{\underline{F}}^G + \hat{\underline{F}}^{\mathrm{ext}} - \hat{\underline{F}}. \tag{71}$$

The mass, gyroscopic, and stiffness matrices of the beam element are

$$\hat{\underline{\underline{M}}} = \int_0^\ell \underline{\underline{N}}^T \underline{\underline{\mathcal{M}}}\, \underline{\underline{N}} \,\mathrm{d}\alpha_1, \tag{72a}$$

$$\hat{\underline{\underline{G}}} = \int_0^\ell \underline{\underline{N}}^T \underline{\underline{\mathcal{G}}}^I \underline{\underline{N}} \,\mathrm{d}\alpha_1, \tag{72b}$$

$$\hat{\underline{\underline{K}}} = \int_0^\ell \left[\underline{\underline{N}}^T (\underline{\underline{\mathcal{K}}}^I + \underline{\underline{\mathcal{Q}}}) \underline{\underline{N}} + \underline{\underline{N}}^T \underline{\underline{\mathcal{P}}}\, \underline{\underline{N}}' + \underline{\underline{N}}'^T \underline{\underline{\mathcal{S}}}\, \underline{\underline{N}}' + \underline{\underline{N}}'^T \underline{\underline{\mathcal{O}}}\, \underline{\underline{N}}\right] \,\mathrm{d}\alpha_1, \tag{72c}$$

respectively, whereas the elemental forces, gravity loads, and externally applied loads are

$$\hat{\underline{F}} = \int_0^\ell \left(\underline{\underline{N}}^T \underline{\mathcal{F}}^I + \underline{\underline{N}}^T \underline{\mathcal{F}}^D + \underline{\underline{N}}'^T \underline{\mathcal{F}}^C\right) \,\mathrm{d}\alpha_1, \tag{73a}$$

$$\hat{\underline{F}}^G = \int_0^\ell \underline{\underline{N}}^T \underline{\mathcal{F}}^G \,\mathrm{d}\alpha_1, \quad \hat{\underline{F}}^{\mathrm{ext}} = \int_0^\ell \underline{\underline{N}}^T \underline{\mathcal{F}}^{\mathrm{ext}} \,\mathrm{d}\alpha_1, \tag{73b}$$

respectively.

8 Interpolation strategy

8.1 Interpolation of the reference and initial configurations

The reference and initial configurations of the system are described by the displacements and rotations at the node of the beam element,

$$\underline{\mathcal{U}}_0^k = \begin{Bmatrix} \underline{u}_0^k \\ \underline{c}_0^k \end{Bmatrix}, \quad \underline{\mathcal{U}}_i^k = \begin{Bmatrix} \underline{u}_i^k \\ \underline{c}_i^k \end{Bmatrix}, \tag{74}$$

respectively, where superscript $(\cdot)^k$ indicates the node number and rotations are represented by the Wiener-Milenković parameters, denoted $\underline{c}$. The displacement field and its spatial derivative are obtain by interpolation as

$$\underline{u}_0(s) = \sum_{k=1}^N h_k\, \underline{u}_0^k, \quad \underline{u}_i(s) = \sum_{k=1}^N h_k\, \underline{u}_i^k, \tag{75a}$$

$$\underline{u}_0'(s) = \sum_{k=1}^N \frac{h_k^+}{J} \underline{u}_0^k, \quad \underline{u}_i'(s) = \sum_{k=1}^N \frac{h_k^+}{J} \underline{u}_i^k, \tag{75b}$$

where N is the total number of nodes of the element and $h_k(s)$ are the shape functions of the element. Notations $(\cdot)'$ $(\cdot)^+$ indicate derivatives with respect to variable x_1 and s, respectively. The Jacobian of the variable transformation is denoted $J = \mathrm{d}\alpha_1/\mathrm{d}s$.

The nodal rotations relative to node 1, denoted $\underline{r}_0^k$ and $\underline{r}_i^k$ in the reference and initial configurations, respectively, are obtained as follows

$$\underline{r}_0^k = \underline{c}_0^{1-} \oplus \underline{c}_0^k, \quad \underline{r}_i^k = \underline{c}_i^{1-} \oplus \underline{c}_i^k, \qquad k = 1, 2, \ldots, N, \tag{76}$$

where symbol $\oplus$ indicates the composition of rotation operation. As discussed by Bauchau *et al.* [23], the relative rotation field is obtained by interpolation of the relative rotations at the nodes,

$$\underline{r}_0(s) = \sum_{k=1}^{N} h_k \, \underline{r}_0^k, \quad \underline{r}_i(s) = \sum_{k=1}^{N} h_k \, \underline{r}_i^k. \tag{77}$$

The rotation field is finally obtained by restoring the rotation of node 1 to find

$$\underline{c}_0(s) = \underline{c}_0^1 \oplus \underline{r}_0(s), \quad \underline{c}_i(s) = \underline{c}_i^1 \oplus \underline{r}_i(s). \tag{78}$$

8.2 Interpolation of the present configuration

At a particular instant in the dynamic simulation, the response of the system is characterized by the nodal displacements, velocities, and accelerations, denoted

$$\underline{\mathcal{U}}^k = \begin{Bmatrix} \underline{u}^k \\ \underline{c}^k \end{Bmatrix}, \quad \underline{\mathcal{V}}^k = \begin{Bmatrix} \underline{v}^k \\ \underline{\omega}^k \end{Bmatrix}, \quad \underline{\mathcal{A}}^k = \begin{Bmatrix} \underline{a}^k \\ \underline{\alpha}^k \end{Bmatrix}. \tag{79}$$

respectively, where $\underline{v}^k = \underline{\dot{u}}^k$ and $\underline{a}^k = \underline{\dot{v}}^f$ are the nodal linear velocities and accelerations, respectively. Array $\underline{\omega}^k$ stores the nodal angular velocities and $\underline{\alpha}^k = \underline{\dot{\omega}}^k$ the nodal angular accelerations.

The displacement, velocity, and acceleration fields are obtained through interpolation as

$$\underline{u}(s) = \sum_{k=1}^{N} h_k \, \underline{u}^k, \quad \underline{v}(s) = \sum_{k=1}^{N} h_k \, \underline{v}^k, \quad \underline{a}(s) = \sum_{k=1}^{N} h_k \, \underline{a}^k. \tag{80}$$

The angular velocity and acceleration fields are obtained in a similar manner as

$$\underline{\omega}(s) = \sum_{k=1}^{N} h_k \, \underline{\omega}^k, \quad \underline{\alpha}(s) = \sum_{k=1}^{N} h_k \, \underline{\alpha}^k. \tag{81}$$

It is also necessary to evaluate the spatial derivatives of the displacements and velocities

$$\underline{u}'(s) = \sum_{k=1}^{N} \frac{h_k^+}{J} \underline{u}^k, \quad \underline{v}'(s) = \sum_{k=1}^{N} \frac{h_k^+}{J} \underline{v}^k, \quad \underline{\omega}'(s) = \sum_{k=1}^{N} \frac{h_k^+}{J} \underline{\omega}^k. \tag{82}$$

The nodal rotations relative to node 1, denoted $\underline{r}^k$, are obtained as follows $\underline{r}^k = \underline{c}^{1-} \oplus \underline{c}^k$, $k = 1, 2, \ldots, N$. The relative rotation field and its spatial derivative are obtained by interpolation of the relative rotations at the nodes,

$$\underline{r}(s) = \sum_{k=1}^{N} h_k \, \underline{r}^k, \quad \underline{r}'(s) = \sum_{k=1}^{N} \frac{h_k^+}{J} \underline{r}^k. \tag{83}$$

The rotation field is finally obtained by restoring the rotation of node 1 to find $\underline{c}(s) = \underline{c}^1 \oplus \underline{r}(s)$.

To evaluate the strain field, the following quantities are required,

$$\underline{E}_1 = \underline{u}_0'(s) + \underline{u}'(s) \tag{84}$$

The curvature vector is found as

$$\underline{\kappa} = \underline{\underline{R}}(\underline{c}^1)\underline{\underline{H}}(\underline{r})\underline{r}'. \tag{85}$$

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