Dymore User's Manual

Formulation and finite element implementation of cable elements

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1 The kinematics of the problem

Consider a cable idealized as a line in space and an inertial reference frame $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ is depicted in fig. 1. A point **P** on the cable is defined by its curvilinear coordinate α_1, which measures length along the reference configuration of the cable. The position vector of point **P** is

$$\underline{x} = \underline{x}(\alpha_1). \tag{1}$$

The base vector in the reference configuration, $\bar{g}_1$, is defined as

$$\bar{g}_1 = \frac{\partial \underline{x}}{\partial \alpha_1}, \tag{2}$$

and clearly is the unit vector tangent to the cable in its reference configuration.

After deformation, the position vector of point **P** is denoted $\underline{X}$ and is written as

$$\underline{X}(\alpha_1) = \underline{x}(\alpha_1) + \underline{u}(\alpha_1), \tag{3}$$

where $\underline{u}$ denote the components of the displacement vector resolved in the inertial basis. The base vector in the deformed configuration becomes

$$\underline{G}_1 = \frac{\partial \underline{X}}{\partial \alpha_1} = \bar{g}_1 + \underline{u}', \tag{4}$$

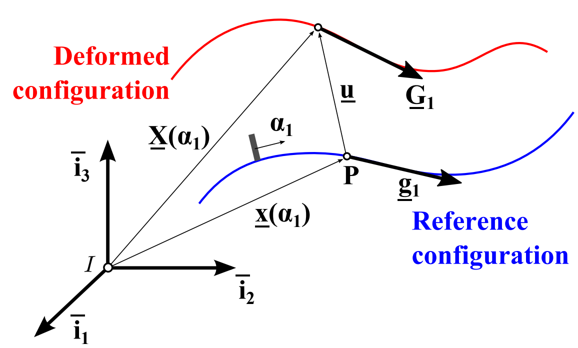

Figure 1: Cable in the reference and deformed configurations

where notation $(\cdot)'$ indicates a derivative with respect to α_1. Note that $\underline{G}_1$ is tangent to the deformed configuration of the cable, but is not a unit vector.

Let $\bar{\jmath}_1$ be a unit vector in the direction of $\underline{G}_1$, then

$$\underline{G}_1 = (1 + \bar{e}_{11})\, \bar{\jmath}_1, \tag{5}$$

where $\bar{e}_{11}$ is a strain related parameter which can be expressed in terms of displacements with the help of eqs. (4) and (5) to find

$$(1 + \bar{e}_{11})^2 = (\bar{g}_1 + \underline{u}')^T (\bar{g}_1 + \underline{u}'). \tag{6}$$

In the present formulation, the axial strain component in the cable is assumed to remain much smaller than unity. In view of eq. (5), the modified deformation gradient tensor then reduces to a single component, $\hat{F}_{11} = 1 + \bar{e}_{11}$, and the Biot strain tensor reduces to a single component,

$$\gamma_{11} = \bar{e}_{11}. \tag{7}$$

This equation gives the physical interpretation of the strain parameter, $\bar{e}_{11}$, as the only non-vanishing component of Biot's strain tensor. This implies that $\bar{e}_{11} \ll 1$ and eq. (6) then yields $(1 + \bar{e}_{11})^2 = (\bar{g}_1 + \underline{u}')^T (\bar{g}_1 + \underline{u}') \approx 1 + 2\bar{e}_{11}$. The strain displacement relationship of the cable is now

$$\gamma_{11} = \bar{e}_{11} \approx \bar{g}_1^T \underline{u}' + \frac{1}{2} \underline{u}'^T \underline{u}', \tag{8}$$

Finally, variation of Biot's strain measure is

$$\delta\gamma_{11} = \delta\underline{u}'^T \left(\bar{g}_1 + \underline{u}'\right). \tag{9}$$

2 Governing equations

The governing equations of the static problem are readily obtained from the principle of virtual work which states

$$-\int_0^\ell \int_\Omega \tau_{11} \delta\gamma_{11} \, \mathrm{d}\Omega \mathrm{d}\alpha_1 + \delta W_{\mathrm{ext}} = 0, \tag{10}$$

where τ_{11} is the single component of the convected Cauchy stress tensor resolved in the material basis, ℓ the length of the cable in the reference configuration, Ω its cross-section area, and δW_{ext} the virtual work done by the externally applied loads. Integrating the left hand side over the cross sectional area of the cable yields $-\int_0^\ell F \delta\gamma_{11} \, \mathrm{d}\alpha_1 + \delta W_{\mathrm{ext}} = 0$, where F is the total axial force in the cable along the material axis $\bar{\jmath}_1$. Introducing the strain variation, eq. (9), then leads to

$$-\int_0^\ell \delta\underline{u}'^T (\bar{g}_1 + \underline{u}') F \, \mathrm{d}\alpha_1 + \int_0^\ell \delta\underline{u}^T \underline{f}_{ext} \, \mathrm{d}\alpha_1 = 0, \tag{11}$$

where $\underline{f}_{ext}$ is the externally applied load per unit length in the reference configuration of the cable. Integration by parts then yields the governing equations of the problem

$$\left[(\bar{g}_1 + \underline{u}')F\right]' = -\underline{f}_{ext}. \tag{12}$$

3 Extension to dynamic problems

In dynamic problems, the inertial forces associated with the motion of the cable can be considered to be externally applied loads, *i.e.*, $\underline{f}_{ext}$ should be replaced by $\underline{f}_{ext} - m\underline{\ddot{u}}$, where m is the mass of the cable per unit length of the reference configuration, $\underline{\ddot{u}}$ the acceleration, and notation $\dot{(\cdot)}$ indicates a derivative with respect to time. The equations of motion then become

$$m\underline{\ddot{u}} - \left[(\bar{g}_1 + \underline{u}')F\right]' = \underline{f}_{ext}. \tag{13}$$

These equations of motion are valid for arbitrarily large displacements of the cable, when the strain component is assumed to remain small.

4 Definition of inertial forces

The inertial forces associated with the cable element are obtained from eqs. (13) as

$$\underline{\mathcal{F}}^I = m\underline{\ddot{u}}. \tag{14}$$

4.1 Linearization of inertial forces

Since the expression for the inertial forces is already linear, the increment of inertial forces is simply

$$\Delta \underline{\mathcal{F}}^I = \underline{\underline{\mathcal{M}}}^I \Delta \underline{\ddot{u}}, \tag{15}$$

where $\mathcal{M}^I$ is the mass matrix associated with the inertial forces, defined as

$$\underline{\underline{\mathcal{M}}}^I = m\underline{\underline{I}}. \tag{16}$$

5 Definition of elastic forces

Linear constitutive laws are assumed for the cable and are written in the following form,

$$F_e = s\gamma_{11}, \tag{17}$$

where F_e is the axial elastic force in the cable and s the cable's axial stiffness. The elastic forces associated with the cable element are then obtained from eqs. (13) as

$$\underline{\mathcal{F}}^C = F_e \underline{E}_1, \tag{18}$$

where $\underline{E}_1 = \bar{g}_1 + \underline{u}'$.

5.1 Linearization of elastic forces

Since the expression for the elastic forces is nonlinear, the computational process will require a linearization. At first, the increment in strain is evaluated to find $\Delta\gamma_{11} = \underline{E}_1^T \Delta \underline{u}'$. Next, the increment in the elastic force is computed

$$\Delta F_e = s\underline{E}_1^T \Delta \underline{u}'. \tag{19}$$

Taking variations of eq. (18) yields the following expression for increments in the elastic forces

$$\Delta \underline{\mathcal{F}}^C = \underline{\underline{\mathcal{S}}} \Delta \underline{u}', \tag{20}$$

where the stiffness matrix, $\underline{\underline{\mathcal{S}}}$, is defined as $\underline{\underline{\mathcal{S}}} = F_e \underline{\underline{I}} + s\underline{E}_1 \underline{E}_1^T$.

6 Definition of dissipative forces

The cable model discussed in the previous section is a purely conservative model, because the elastic force are proportional the strain measures. It is often desirable to also introduce dissipative forces in the cable model. By analogy to eq. (17), the dissipative force will be written as

$$F_d = \mu s \dot{\gamma}_{11}. \tag{21}$$

where μ is the damping coefficient of units 1/sec, and $\dot{\gamma}_{11}$ the time rate of change of Biot's strain. Since the dissipative mechanisms in the cable are not well understood, it is postulated that the damping coefficient is proportional to the cable's axial stiffness and is written as μs. The time rate of change of Biot's strain is readily obtained from eq. (8) as

$$\dot{\gamma}_{11} = \underline{E}_1^T \underline{\dot{u}}'. \tag{22}$$

The dissipative forces, $\underline{\mathcal{F}}^{dC}$, associated with the cable element are,

$$\underline{\mathcal{F}}^{dC} = F_d \underline{E}_1. \tag{23}$$

6.1 Linearization of dissipative forces

Since the expression for the dissipative forces is nonlinear, the computational process will require a linearization. At first, the increment in the time rate of change of the strain is evaluated to find

$$\Delta\dot{\gamma}_{11} = \underline{E}_1^T \Delta \underline{\dot{u}}' + \underline{\dot{u}}'^T \Delta \underline{u}'. \tag{24}$$

Next, the increment in the dissipative forces is computed $\Delta F_d = \mu s \left(\underline{\dot{u}}'^T \Delta \underline{u}' + \underline{E}_1^T \Delta \underline{\dot{u}}' \right)$. Taking variations of eq. (23) yields the following expression for increments in the dissipative forces

$$\Delta \underline{\mathcal{F}}^{dC} = \underline{\underline{\mathcal{S}}}^d \Delta \underline{u}' + \underline{\underline{\mathcal{E}}}^d \Delta \underline{\dot{u}}', \tag{25}$$

where $\underline{\underline{\mathcal{S}}}^d = F_d \underline{\underline{I}} + \mu s \underline{E}_1 \underline{\dot{u}}'^T$ and $\underline{\underline{\mathcal{E}}}^d = \mu s \underline{E}_1 \underline{E}_1^T$.

7 Gravity forces for cables

Gravity forces will be applied on cable due to their mass distribution. The potential of the gravity forces is written as

$$V = -m\underline{g}^T(\underline{x}+\underline{u}). \tag{26}$$

A variation of this potential is written as $\delta V = -m\underline{g}^T\delta\underline{u}$. The gravity forces acting on the material points of cable are readily found as

$$\underline{\mathcal{F}}^G = m\underline{g}. \tag{27}$$

8 Static formulation of cables

It is sometime desirable to perform a static analysis, *i.e.*, an analysis that neglects inertial forces. This can be easily achieved setting all time derivatives to zero in the equations of motion derived in the previous paragraphs, and solve the resulting nonlinear algebraic system. Furthermore, it is also possible to perform an eigenvalue analysis to determine the natural frequencies of the structure. In that case, the mass, gyroscopic, and stiffness matrices of the system must be evaluated.

8.1 Rotating structure

An interesting case to consider is that where one of more subcomponents of the structure to be analyzed are rotating at a constant angular velocity with respect to other subcomponents. While such problem is not strictly speaking a static problem because the rotating subcomponents feature non-vanishing linear and angular velocities, it can be formulated as a static problem where the centrifugal forces generated by the rigid body rotation are applied on the otherwise static structure.

Figure 2 shows a non-rotating structure, a simple cantilevered beam in this case, with a rotating subcomponent, a tip rotating beam. Frame $\mathcal{F}^I = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$ is the inertial frame, frame $\mathcal{F}_0^A = [\mathbf{A}, \mathcal{B}_0^A = (\bar{b}_{10}^A, \bar{b}_{20}^A, \bar{b}_{30}^A)]$ is attached to the non-rotating structure at point $\mathbf{A}$ in its reference configuration, and frame $\mathcal{F}^A = [\mathbf{A}, \mathcal{B}^A = (\bar{b}_1^A, \bar{b}_2^A, \bar{b}_3^A)]$ is attached to the non-rotating structure at point $\mathbf{A}$ in the deformed configuration. Frame $\mathcal{F}^R = [\mathbf{A}, \mathcal{B}^R = (\bar{b}_1^R, \bar{b}_2^R, \bar{b}_3^R)]$ rotates at a constant angular velocity, $\underline{\omega}_r$, with respect to frame $\mathcal{F}^A$.

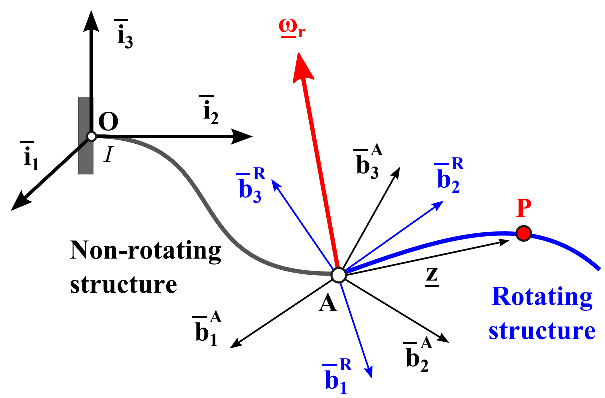

Figure 2: A structure featuring a subcomponent rotating at a constant angular velocity $\underline{\omega}_r$.

Let $\underline{\underline{R}}_0^A$ and $\underline{\underline{R}}^A$ be the components of the rotation tensors that bring triad $\mathcal{I}$ to $\mathcal{B}_0^A$ and triad $\mathcal{B}_0^A$ to $\mathcal{B}^A$, respectively, both resolved in the inertial basis, $\mathcal{I}$. Furthermore, let $\underline{\underline{R}}_r^*$ be the components of the rotation tensor describing the finite rotation from triad $\mathcal{B}^A$ to $\mathcal{B}^R$, resolved in basis $\mathcal{B}^A$. With these definition, it follows that

$$\bar{b}_1^R = (\underline{\underline{R}}^A \underline{\underline{R}}_0^A)\underline{\underline{R}}_r^* \bar{\imath}_1. \tag{28}$$

The time derivative of this unit vector is readily computed as

$$\dot{\bar{b}}_1^R = \widetilde{\omega}_r \bar{b}_1^R, \tag{29}$$

where $\widetilde{\omega}_r^* = \dot{\underline{\underline{R}}}_r^* \underline{\underline{R}}_r^{*T}$ are the components of the relative angular velocity of the rotating subcomponent with respect to its non-rotating counterpart resolved in basis $\mathcal{B}^A$, and $\underline{\omega}_r = (\underline{\underline{R}}^A \underline{\underline{R}}_0^A)\underline{\omega}_r^*$ the components of the same vector resolved in the inertial basis. It is assumed here that the rotating subcomponent rotates at a given, constant angular velocity with respect to the non-rotating subcomponent, *i.e.*, $\underline{\omega}_r^*$ is a constant vector in time. Note that for a static problem, components $\underline{\omega}_r$ are also constant, although not known, because the motion of frame $\mathcal{F}^A$ is itself unknown.

8.2 Rigid body rotation forces

Consider now an arbitrary point $\mathbf{P}$ of the rotating structure and its position vector, $\underline{z}$, with respect to point $\mathbf{A}$, as shown in fig. 2. Since the motion of the rotating structure is a rigid body rotation, the inertial velocity of point $\mathbf{P}$ is simply $\dot{\underline{u}} = \widetilde{\omega}_r \underline{z}$. The components of this vector in the rotating frame, $\dot{\underline{u}}^* = \underline{\underline{R}}_r^{*T}(\underline{\underline{R}}^A \underline{\underline{R}}_0^A)^T \dot{\underline{u}}$, are constant because the motion of the rotating structure is at a constant angular velocity, $\underline{\omega}_r^*$. This implies $R_r^{*T}\widetilde{\omega}_r^*(\underline{\underline{R}}^A\underline{\underline{R}}_0^A)^T\underline{\dot{u}}+\underline{\underline{R}}_r^{*T}(\underline{\underline{R}}^A\underline{\underline{R}}_0^A)^T\underline{\ddot{u}}=0$, and hence $\underline{\ddot{u}}=\widetilde{\omega}_r\underline{\dot{u}}$. In summary, the velocity and acceleration fields of the rotating structure are

$$\begin{Bmatrix}\underline{\dot{u}}\\ \underline{\omega}\end{Bmatrix}=\begin{Bmatrix}\widetilde{\omega}_r\underline{z}\\ \underline{\omega}_r\end{Bmatrix},\quad \text{and}\quad \begin{Bmatrix}\underline{\ddot{u}}\\ \underline{\dot{\omega}}\end{Bmatrix}=\begin{Bmatrix}\widetilde{\omega}_r\widetilde{\omega}_r\underline{z}\\ 0\end{Bmatrix},\tag{30}$$

respectively.

The forces associated with the rigid body rotation are readily found by inserting these fields into eq. (14) to find the centrifugal forces applied to the rotating structure as

$$\underline{\mathcal{F}}^I=m\widetilde{\omega}_r\widetilde{\omega}_r\underline{z}.\tag{31}$$

Note that these are the "static forces," *i.e.*, time independent forces, applied to the structure due to its rigid body rotation. In a static analysis, the steady centrifugal forces given by eq. (31) are applied to the non-rotating structure. In other words, for the static analysis, the structure is not rotating, but subjected to the steady centrifugal loading that would appear *if the structure were actually rotating*.

8.3 Linearization of the rigid body rotation forces

Although the rigid body rotation considered here is characterized by an angular velocity vector of constant magnitude, the orientation of this angular velocity vector is unknown as the structure is allowed to deform statically, *i.e.*, the rotation tensor at point **A** is unknown. Similarly, the relative position vector $\underline{z}$ shown in fig. 2 is not known because it is a function of the displacements at points **P** and **A**. These facts are expressed by the following relationships,

$$\Delta\underline{\omega}_r=\widetilde{\omega}_r^T\underline{\Delta\psi}_A,\tag{32a}$$

$$\Delta\underline{z}=\Delta\underline{u}-\Delta\underline{u}_A,\tag{32b}$$

where $\Delta\underline{u}_A$ and $\underline{\Delta\psi}_A$ are small increments in the position and rotation at point **A**.

To solve the nonlinear static problem, a linearization process similar to that described in section 4.1 must be performed. The linearization of the inertial forces associated with the rigid body motion can be cast as

$$\Delta\underline{\mathcal{F}}_r=m\left[\Delta\underline{\ddot{u}}+\Delta(\widetilde{\omega}_r\widetilde{\omega}_r\underline{z})\right].\tag{33}$$

The second term appearing in the parenthesis of this equation corresponds to perturbations in the rigid body rotation. With the help of eqs. (32), these perturbations are expressed as

$$\Delta(\widetilde{\omega}_r\widetilde{\omega}_r\underline{z})=\widetilde{\omega}_r\widetilde{\omega}_r\Delta\underline{u}+\widetilde{\omega}_r\widetilde{\omega}_r^T\Delta\underline{u}_A+(\widetilde{z}\widetilde{\omega}_r-2\widetilde{\omega}_r\widetilde{z})\widetilde{\omega}_r^T\underline{\Delta\psi}_A.\tag{34}$$

Introducing this result into eq. (33) then yields the desired linearization of the centrifugal forces as

$$\Delta\underline{\mathcal{F}}_r=m\Delta\underline{\ddot{u}}+\underline{\underline{\mathcal{K}}}_r\Delta\underline{u}+\underline{\underline{\mathcal{K}}}_A^u\Delta\underline{u}_A+\underline{\underline{\mathcal{K}}}_A^c\underline{\Delta\psi}_A,\tag{35}$$

where

$$\underline{\underline{\mathcal{K}}}_r=m\widetilde{\omega}_r\widetilde{\omega}_r,\quad \underline{\underline{\mathcal{K}}}_A^u=m\widetilde{\omega}_r\widetilde{\omega}_r^T,\quad \underline{\underline{\mathcal{K}}}_A^c=m(\widetilde{z}\widetilde{\omega}_r-2\widetilde{\omega}_r\widetilde{z})\widetilde{\omega}_r^T.\tag{36}$$

9 Finite element formulation of cables

With the notation defined in eqs. (14), (18) and (23), the equations of motion of cable, eqs. (13), can be recast in the following form

$$\underline{\mathcal{F}}^I-\left(\underline{\mathcal{F}}^C+\underline{\mathcal{F}}^{dC}\right)'=\underline{\mathcal{F}}^G+\underline{\mathcal{F}}^{\text{ext}},\tag{37}$$

where $\underline{\mathcal{F}}^{\text{ext}}$ are the external forces applied to the cable. A weighted residual formulation will be used here to enforce these dynamic equilibrium conditions

$$\int_0^\ell \underline{\underline{N}}^T\left[\underline{\mathcal{F}}^I-\left(\underline{\mathcal{F}}^C+\underline{\mathcal{F}}^{dC}\right)'-\underline{\mathcal{F}}^G-\underline{\mathcal{F}}^{\text{ext}}\right]\mathrm{d}\alpha_1=0,\tag{38}$$

where ℓ is the length of the cable element and $W(\alpha_1)$ a matrix storing the selected test functions. An integration by parts is performed on the second term of this equation, leading to

$$\int_0^\ell\left[\underline{\underline{N}}^T\underline{\mathcal{F}}^I+W'^T\left(\underline{\mathcal{F}}^C+\underline{\mathcal{F}}^{dC}\right)\right]\mathrm{d}\alpha_1=\int_0^\ell\underline{\underline{N}}^T(\underline{\mathcal{F}}^G+\underline{\mathcal{F}}^{\text{ext}})\,\mathrm{d}\alpha_1.\tag{39}$$

Since this set of algebraic equations is nonlinear, a linearization process is required to solve it. Eqs. (15), (19) and (25) are introduced to find

$$\int_0^\ell \left[\underline{\underline{N}}^T\left(\underline{\mathcal{F}}^I + \mathcal{M}^I\Delta\underline{\ddot{u}}\right) + \underline{\underline{N}}'^T\left(\underline{\mathcal{F}}^C + \mathcal{S}\Delta\underline{u}' + \underline{\mathcal{F}}^{dC} + \mathcal{S}^d\Delta\underline{u}' + \mathcal{E}^d\Delta\underline{\dot{u}}'\right)\right]\,\mathrm{d}\alpha_1 = \int_0^\ell \underline{\underline{N}}^T(\underline{\mathcal{F}}^G + \underline{\mathcal{F}}^{\text{ext}})\,\mathrm{d}\alpha_1 \tag{40}$$

Next, the elemental displacement, velocity and acceleration fields are expressed in terms of their nodal values using the assumed shape functions, $W(\alpha_1)$,

$$\underline{u}(\alpha_1) = \underline{\underline{N}}(\alpha_1)\underline{\hat{u}}, \quad \underline{u}'(\alpha_1) = \underline{\underline{N}}'(\alpha_1)\underline{\hat{u}}, \quad \underline{\dot{u}}'(\alpha_1) = \underline{\underline{N}}'(\alpha_1)\underline{\dot{\hat{u}}}, \quad \underline{\ddot{u}}(\alpha_1) = \underline{\underline{N}}(\alpha_1)\underline{\ddot{\hat{u}}}, \tag{41}$$

where $\underline{\hat{u}}$ and $\underline{\ddot{\hat{u}}}$ are the nodal displacements and accelerations, $\underline{\hat{u}}'$ and $\underline{\dot{\hat{u}}}'$ the spatial derivatives of nodal displacements and velocities, respectively. With the help of these interpolations of elemental fields, the weak statement of dynamic equilibrium becomes

$$\underline{\underline{\hat{M}}}\Delta\underline{\ddot{\hat{u}}} + \underline{\underline{\hat{G}}}\Delta\underline{\dot{\hat{u}}} + \underline{\underline{\hat{K}}}\Delta\underline{\hat{u}} = \underline{\hat{F}}^G + \underline{\hat{F}}^{\text{ext}} - \underline{\hat{F}}. \tag{42}$$

The mass, gyroscopic and stiffness matrices of the cable element are

$$\underline{\underline{\hat{M}}} = \int_0^\ell \underline{\underline{N}}^T\underline{\underline{\mathcal{M}}}^I\underline{\underline{N}}\,\mathrm{d}\alpha_1, \tag{43a}$$

$$\underline{\underline{\hat{G}}} = \int_0^\ell \underline{\underline{N}}'^T\underline{\underline{\mathcal{E}}}^d\underline{\underline{N}}'\,\mathrm{d}\alpha_1, \tag{43b}$$

$$\underline{\underline{\hat{K}}} = \int_0^\ell \underline{\underline{N}}'^T\left(\underline{\underline{\mathcal{S}}} + \underline{\underline{\mathcal{S}}}^d\right)\underline{\underline{N}}'\,\mathrm{d}\alpha_1, \tag{43c}$$

respectively, whereas the elemental forces, gravity loads, and externally applied loads are

$$\underline{\hat{F}} = \int_0^\ell \left[\underline{\underline{N}}^T\underline{\mathcal{F}}^I + \underline{\underline{N}}'^T\left(\underline{\mathcal{F}}^C + \underline{\mathcal{F}}^{dC}\right)\right]\,\mathrm{d}\alpha_1, \tag{44a}$$

$$\underline{\hat{F}}^G = \int_0^\ell \underline{\underline{N}}^T\underline{\mathcal{F}}^G\,\mathrm{d}\alpha_1, \quad \underline{\hat{F}}^{\text{ext}} = \int_0^\ell \underline{\underline{N}}^T\underline{\mathcal{F}}^{\text{ext}}\,\mathrm{d}\alpha_1, \tag{44b}$$

respectively.