Dymore User's Manual

Formulation and finite element implementation of flexible joints

Contents

1 Formulation of the flexible joint

The flexible joint consists of a set of six concentrated springs and dampers connecting two bodies. The relative motion of the two bodies consists of relative displacements of two points of the bodies and relative rotations of the two bodies. The relative displacements stretch the first three springs, also referred to as *rectilinear springs*, whereas the relative rotations stretch the next three springs, referred to as *rotational springs*. Similarly, the rates of change of the relative displacements stroke the first three dampers, also referred to as *rectilinear dampers*, whereas the rates of change of the relative rotations stroke the next three dampers, referred to as *rotational dampers*. If any of the spring constants are select to be large values, the corresponding relative motion is driven to become very small, and the flexible joint behaves like a lower pair joint. This observation clearly underlines the close relationship between the flexible joint and the lower pair joints.

Consider two rigid bodies linked together by rectilinear and torsional springs and dampers at a point, as depicted in fig. 1. The relative displacements between point **K** and **L**, denoted $\underline{u}$, is simply $\underline{u} = \underline{u}^\ell - \underline{u}^k$. The components of this vector, resolved in basis in $\mathcal{B}^k$, are then

$$\underline{u}^* = (\underline{\underline{R}}^k \underline{\underline{R}}_0)^T (\underline{u}^\ell - \underline{u}^k) = (\underline{\underline{R}}^k \underline{\underline{R}}_0)^T \underline{u}. \tag{1}$$

The relative rotation of the rigid bodies will be measured by the following vector

$$\underline{s}^* = \frac{1}{2} \begin{Bmatrix} g_{32} - g_{23} \\ g_{13} - g_{31} \\ g_{21} - g_{12} \end{Bmatrix}, \tag{2}$$

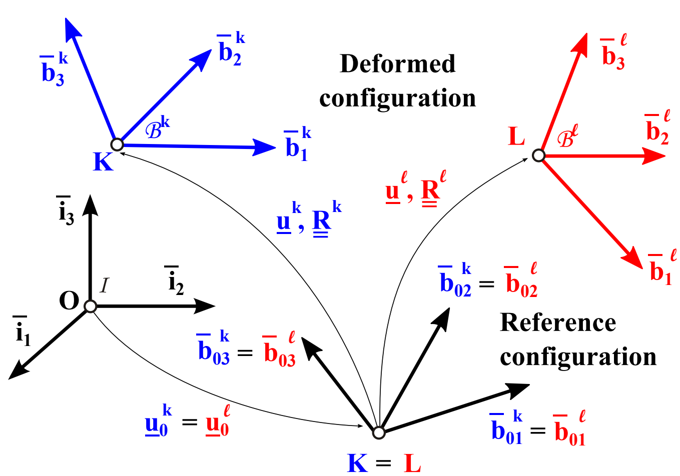

Figure 1: Configuration of a flexible joint.

where the scalars $g_{\alpha\beta} = \bar{b}_\alpha^{kT} \bar{b}_\beta^\ell$. It is clear that $\bar{b}_\alpha^k = (\underline{\underline{R}}^k \underline{\underline{R}}_0)\,\bar{\imath}_\alpha$ and $\bar{b}_\alpha^\ell = (\underline{\underline{R}}^\ell \underline{\underline{R}}_0)\bar{\imath}_\alpha$. Hence, the components of the relative rotation tensor that brings basis $\mathcal{B}^k$ to basis $\mathcal{B}^\ell$, resolved in the inertial frame, are $\underline{\underline{R}} = \underline{\underline{R}}^\ell \underline{\underline{R}}^{kT}$. The components of the same rotation tensor resolved in basis $\mathcal{B}^k$ are then $\underline{\underline{R}}^* = \underline{\underline{R}}_0^T \underline{\underline{R}}^{kT} \underline{\underline{R}}^\ell \underline{\underline{R}}_0$. Consequently, the measure of the relative rotation of the rigid bodies defined eq. (2) becomes

$$\underline{s}^* = \frac{1}{2} \begin{Bmatrix} R_{32}^* - R_{23}^* \\ R_{13}^* - R_{31}^* \\ R_{21}^* - R_{12}^* \end{Bmatrix} = \text{axial}(\underline{\underline{R}}^*). \tag{3}$$

If $\hat{e}^*$ are the Euler parameters of relative rotation tensor $\underline{\underline{R}}^*$, it follows that

$$\underline{s}^* = 2 \begin{Bmatrix} e_0^* e_1^* \\ e_0^* e_2^* \\ e_0^* e_3^* \end{Bmatrix} = 2 e_0^* \, \underline{e}^* = 2 \cos\frac{\phi}{2} \sin\frac{\phi}{2}\, \bar{n}^* = \sin\phi\, \bar{n}^* \tag{4}$$

where ϕ the magnitude of the relative rotation and $\bar{n}^*$ the components of the unit vector about which it takes place resolved in basis $\mathcal{B}^k$. This result provides the physical interpretation of vector $\underline{s}^*$: it represents the linear parameters of the relative rotation from basis $\mathcal{B}^k$ to basis $\mathcal{B}^\ell$, resolved in $\mathcal{B}^k$.

1.1 Elastic forces in the flexible joint

The deformation of the flexible joint will be characterized by the following strain measures

$$\underline{\epsilon}^* = \begin{Bmatrix} \underline{u}^* \\ \underline{s}^* \end{Bmatrix}, \tag{5}$$

and the following expression is assumed for the strain energy in the flexible joint

$$A = \frac{1}{2}\, \underline{\epsilon}^{*T} \underline{\underline{\mathcal{D}}}^* \underline{\epsilon}^*, \tag{6}$$

where $\underline{\underline{\mathcal{D}}}^*$ are the components of the flexible joint stiffness matrix resolved in the the body attached basis $\mathcal{B}^k$. This 6×6 stiffness matrix is fully populated allowing the modeling of the various linear and torsional stiffnesses, as well as potential elastic couplings. Virtual changes in the strain energy now become

$$\delta A = \delta \underline{\epsilon}^{*T} \underline{\underline{\mathcal{D}}}^* \underline{\epsilon}^* = \lfloor \delta \underline{u}^{*T}, \delta \underline{s}^{*T} \rfloor \begin{Bmatrix} \underline{f}^* \\ \underline{m}^* \end{Bmatrix}, \tag{7}$$

where the forces and moments in the flexible joint were defined as

$$\begin{Bmatrix} \underline{f}^* \\ \underline{m}^* \end{Bmatrix} = \underline{\underline{\mathcal{D}}}^* \underline{\epsilon}^*. \tag{8}$$

Virtual changes in the flexible joint deformation measure, $\delta \underline{\epsilon}^*$, can be readily evaluated from eq. (5) as

$$\delta \underline{\epsilon}^* = \begin{bmatrix} (\underline{\underline{R}}^k \underline{\underline{R}}_0)^T & 0 \\ 0 & \underline{\underline{S}}^T \end{bmatrix} \left\{ - \begin{bmatrix} \underline{\underline{I}} & \widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix} \begin{Bmatrix} \delta \underline{u}^k \\ \delta \underline{\psi}^k \end{Bmatrix} + \begin{Bmatrix} \delta \underline{u}^\ell \\ \delta \underline{\psi}^\ell \end{Bmatrix} \right\}. \tag{9}$$

The operator $\underline{\underline{S}}$ is defined as

$$\underline{\underline{S}} = \frac{1}{2} \left[\underline{h}_{23} - \underline{h}_{32},\ \underline{h}_{31} - \underline{h}_{13},\ \underline{h}_{12} - \underline{h}_{21} \right], \tag{10}$$

where the vectors $\underline{h}_{\alpha\beta} = \widetilde{b}_\alpha^k \bar{b}_\beta^\ell$. Virtual changes in the strain energy now become

$$\delta A = \left\{ - \lfloor \delta \underline{u}^{kT}, \delta \underline{\psi}^{kT} \rfloor \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{0}} \\ \widetilde{u} & \underline{\underline{I}} \end{bmatrix} + \lfloor \delta \underline{u}^{\ell T}, \delta \underline{\psi}^{\ell T} \rfloor \right\} \begin{Bmatrix} \underline{f} \\ \underline{m} \end{Bmatrix}, \tag{11}$$

where $\underline{f} = (\underline{\underline{R}}^k \underline{\underline{R}}_0) \underline{f}^*$ and $\underline{m} = \underline{\underline{S}}\, \underline{m}^*$.

The expression for the elastic forces, $\underline{\mathcal{F}}_e$, in the flexible joint are now

$$\delta A = \begin{Bmatrix} \delta \underline{u}^k \\ \delta \underline{\psi}^k \\ \delta \underline{u}^\ell \\ \delta \underline{\psi}^\ell \end{Bmatrix}^T \begin{Bmatrix} -\underline{f} \\ -\underline{m} - \widetilde{u} \underline{f} \\ \underline{f} \\ \underline{m} \end{Bmatrix} = \begin{Bmatrix} \delta \underline{u}^k \\ \delta \underline{\psi}^k \\ \delta \underline{u}^\ell \\ \delta \underline{\psi}^\ell \end{Bmatrix}^T \underline{\mathcal{F}}_e. \tag{12}$$

The following compact notation is used to express these forces

$$\underline{\mathcal{F}}_e = \begin{Bmatrix} \underline{F}_e^k \\ \underline{F}_e^\ell \end{Bmatrix}; \quad \underline{F}_e^\ell = \begin{Bmatrix} \underline{f} \\ \underline{m} \end{Bmatrix}; \quad \underline{F}_e^k = - \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{0}} \\ \widetilde{u} & \underline{\underline{I}} \end{bmatrix} \underline{F}_e^\ell. \tag{13}$$

1.1.1 Linearization of the elastic forces

The solution process will require a linearization of the elastic forces. At first, the following increments are computed

$$\Delta \underline{f} = \Delta(\underline{\underline{R}}^k \underline{\underline{R}}_0 \underline{f}^*) = \widetilde{f}^T \Delta \underline{\psi}^k + (\underline{\underline{R}}^k \underline{\underline{R}}_0) \Delta \underline{f}^*, \quad \Delta \underline{m} = \Delta(\underline{\underline{S}}\, \underline{m}^*) = \underline{\underline{X}}^k \Delta \underline{\psi}^k + \underline{\underline{X}}^\ell \Delta \underline{\psi}^\ell + \underline{\underline{S}} \Delta \underline{m}^*.$$

In these expressions, the following notation was used

$$\underline{\underline{G}}^k = \left[\underline{g}_1^k, \underline{g}_2^k, \underline{g}_3^k \right] = (\underline{\underline{R}}^k \underline{\underline{R}}_0) \frac{\widetilde{m}^*}{2}, \quad \underline{\underline{G}}^\ell = \left[\underline{g}_1^\ell, \underline{g}_2^\ell, \underline{g}_3^\ell \right] = (\underline{\underline{R}}^\ell \underline{\underline{R}}_0) \frac{\widetilde{m}^*}{2},$$

and

$$\underline{\underline{X}}^k(\underline{m}^*) = \widetilde{g}_1^\ell \widetilde{b}_1^k + \widetilde{g}_2^\ell \widetilde{b}_2^k + \widetilde{g}_3^\ell \widetilde{b}_3^k, \quad \underline{\underline{X}}^\ell(\underline{m}^*) = \widetilde{g}_1^k \widetilde{b}_1^\ell + \widetilde{g}_2^k \widetilde{b}_2^\ell + \widetilde{g}_3^k \widetilde{b}_3^\ell.$$

Next, the linearization of the elastic force components resolved in the material basis is computed with the help of eq. (9) to find

$$\Delta \begin{Bmatrix} \underline{f}^* \\ \underline{m}^* \end{Bmatrix} = \underline{\underline{\mathcal{D}}}^* \begin{bmatrix} (\underline{\underline{R}}^k \underline{\underline{R}}_0)^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}}^T \end{bmatrix} \left\{ - \begin{bmatrix} \underline{\underline{I}} & \widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix} \begin{Bmatrix} \Delta \underline{u}^k \\ \Delta \underline{\psi}^k \end{Bmatrix} + \begin{Bmatrix} \Delta \underline{u}^\ell \\ \Delta \underline{\psi}^\ell \end{Bmatrix} \right\}.$$

Combining the above results yields

$$\Delta \underline{F}_e^\ell = \Delta \begin{Bmatrix} \underline{f} \\ \underline{m} \end{Bmatrix} = \underline{\underline{K}}^{\ell k} \begin{Bmatrix} \Delta \underline{u}^k \\ \Delta \underline{\psi}^k \end{Bmatrix} + \underline{\underline{K}}^{\ell\ell} \begin{Bmatrix} \Delta \underline{u}^\ell \\ \Delta \underline{\psi}^\ell \end{Bmatrix},$$

where

$$\underline{\underline{K}}^{\ell k} = \left\{ \begin{bmatrix} \underline{\underline{0}} & \widetilde{f}^T \\ \underline{\underline{0}} & \underline{\underline{X}}^k \end{bmatrix} - \underline{\underline{\mathcal{D}}} \begin{bmatrix} \underline{\underline{I}} & \widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix} \right\}, \quad \underline{\underline{K}}^{\ell\ell} = \left\{ \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{X}}^\ell \end{bmatrix} + \underline{\underline{\mathcal{D}}} \right\}.$$

The components of the stiffness matrix resolved in the inertial basis are

$$\underline{\underline{\mathcal{D}}} = \begin{bmatrix} (\underline{\underline{R}}^k \underline{\underline{R}}_0) & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}} \end{bmatrix} \underline{\underline{\mathcal{D}}}^* \begin{bmatrix} (\underline{\underline{R}}^k \underline{\underline{R}}_0)^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}}^T \end{bmatrix}.$$

The linearization of the second part of the elastic forces is

$$\Delta \underline{F}_e^k = \underline{\underline{K}}^{kk} \begin{Bmatrix} \Delta \underline{u}^k \\ \Delta \underline{\psi}^k \end{Bmatrix} + \underline{\underline{K}}^{k\ell} \begin{Bmatrix} \Delta \underline{u}^\ell \\ \Delta \underline{\psi}^\ell \end{Bmatrix},$$

where

$$\underline{\underline{K}}^{kk} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \widetilde{f}^T & \underline{\underline{0}} \end{bmatrix} - \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{0}} \\ \widetilde{u} & \underline{\underline{I}} \end{bmatrix} \underline{\underline{K}}^{\ell k}, \quad \underline{\underline{K}}^{k\ell} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \widetilde{f} & \underline{\underline{0}} \end{bmatrix} - \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{0}} \\ \widetilde{u} & \underline{\underline{I}} \end{bmatrix} \underline{\underline{K}}^{\ell\ell}.$$

1.2 Dissipative forces in the flexible joint

The rate of deformation of the flexible joint will be characterized by the following strain rate measures

$$\underline{\dot{\epsilon}}^* = \begin{Bmatrix} \underline{\dot{u}}^* \\ \underline{\dot{s}}^* \end{Bmatrix}. \tag{14}$$

By analogy with eq. (8), the dissipative forces and moments in the flexible joint are assumed to take the following form

$$\begin{Bmatrix} \underline{h}^* \\ \underline{g}^* \end{Bmatrix} = \underline{\underline{\mathcal{Q}}}^* \underline{\dot{\epsilon}}^*, \tag{15}$$

where $\underline{\underline{\mathcal{Q}}}^*$ are the components of the flexible joint damping matrix resolved in the the body attached basis $\mathcal{B}^k$. This 6×6 damping matrix is fully populated allowing the modeling of the various rectilinear and torsional damping coefficients, as well as potential couplings.

The strain rates in the flexible joint readily follow from eq. (5) as

$$\underline{\dot{\epsilon}}^* = \begin{bmatrix} (\underline{\underline{R}}^k \underline{\underline{R}}_0)^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}}^T \end{bmatrix} \left\{ - \begin{bmatrix} \underline{\underline{I}} & \widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix} \begin{Bmatrix} \underline{\dot{u}}^k \\ \underline{\omega}^k \end{Bmatrix} + \begin{Bmatrix} \underline{\dot{u}}^\ell \\ \underline{\omega}^\ell \end{Bmatrix} \right\} = \begin{bmatrix} (\underline{\underline{R}}^k \underline{\underline{R}}_0)^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}}^T \end{bmatrix} \begin{Bmatrix} \underline{\dot{u}} + \widetilde{u} \underline{\omega}^k \\ \underline{\omega}^\ell - \underline{\omega}^k \end{Bmatrix}, \tag{16}$$

where operator $\underline{\underline{S}}$ was defined in eq. (10) and the angular velocities of bodies k and ℓ are denoted $\underline{\omega}^k$ and $\underline{\omega}^\ell$, respectively. The virtual work done by the dissipative forces and moments is

$$\delta W = \left\{ -\lfloor \delta \underline{u}^{kT}, \delta \underline{\psi}^{kT} \rfloor \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{0}} \\ \widetilde{u} & \underline{\underline{I}} \end{bmatrix} + \lfloor \delta \underline{u}^{\ell T}, \delta \underline{\psi}^{\ell T} \rfloor \right\} \begin{Bmatrix} \underline{h} \\ \underline{g} \end{Bmatrix}, \tag{17}$$

where $\underline{h} = (\underline{\underline{R}}^k \underline{\underline{R}}_0) \underline{h}^*$ and $\underline{g} = \underline{\underline{S}}\, \underline{g}^*$. The expression for the dissipative forces, $\underline{\mathcal{F}}_d$, in the flexible joint now becomes

$$\delta W = \begin{Bmatrix} \delta \underline{u}^k \\ \delta \underline{\psi}^k \\ \delta \underline{u}^l \\ \delta \underline{\psi}^l \end{Bmatrix}^T \begin{Bmatrix} -\underline{h} \\ -\underline{g} - \widetilde{u} \underline{h} \\ \underline{h} \\ \underline{g} \end{Bmatrix} = \begin{Bmatrix} \delta \underline{u}^k \\ \delta \underline{\psi}^k \\ \delta \underline{u}^l \\ \delta \underline{\psi}^l \end{Bmatrix}^T \underline{\mathcal{F}}_d. \tag{18}$$

The following compact notation is used to express these forces

$$\underline{\mathcal{F}}_d = \begin{Bmatrix} \underline{F}_d^k \\ \underline{F}_d^\ell \end{Bmatrix}; \quad \underline{F}_d^\ell = \begin{Bmatrix} \underline{h} \\ \underline{g} \end{Bmatrix}; \quad \underline{F}_d^k = - \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{0}} \\ \widetilde{u} & \underline{\underline{I}} \end{bmatrix} \underline{F}_d^\ell. \tag{19}$$

1.2.1 Linearization of the dissipative forces

The solution process will require a linearization of the dissipative forces. At first, the following increments are computed

$$\Delta \underline{h} = \Delta(\underline{\underline{R}}^k \underline{\underline{R}}_0 \underline{h}^*) = \widetilde{h}^T \underline{\Delta\psi}^k + (\underline{\underline{R}}^k \underline{\underline{R}}_0)\Delta \underline{h}^*, \quad \Delta \underline{g} = \Delta(\underline{\underline{S}}\,\underline{g}^*) = \underline{\underline{Y}}^k \underline{\Delta\psi}^k + \underline{\underline{Y}}^\ell \underline{\Delta\psi}^\ell + \underline{\underline{S}}\,\Delta \underline{g}^*.$$

In these expression, the following notation was used

$$\underline{\underline{G}}^k = \left[\underline{g}_1^k, \underline{g}_2^k, \underline{g}_3^k\right] = (\underline{\underline{R}}^k \underline{\underline{R}}_0)\frac{\widetilde{g}^*}{2}, \quad \underline{\underline{G}}^\ell = \left[\underline{g}_1^\ell, \underline{g}_2^\ell, \underline{g}_3^\ell\right] = (\underline{\underline{R}}^\ell \underline{\underline{R}}_0)\frac{\widetilde{g}^*}{2},$$

and

$$\underline{\underline{Y}}^k(\underline{g}^*) = \widetilde{g}_1^\ell \widetilde{b}_1^k + \widetilde{g}_2^\ell \widetilde{b}_2^k + \widetilde{g}_3^\ell \widetilde{b}_3^k, \quad \underline{\underline{Y}}^\ell(\underline{g}^*) = \widetilde{g}_1^k \widetilde{b}_1^\ell + \widetilde{g}_2^k \widetilde{b}_2^\ell + \widetilde{g}_3^k \widetilde{b}_3^\ell.$$

Next, the linearization of the components of the dissipative forces resolved in the material basis are computed with the help of eq. (16) to find

$$\Delta \begin{Bmatrix} \underline{h}^* \\ \underline{g}^* \end{Bmatrix} = \underline{\underline{Q}}^* \begin{bmatrix} (\underline{\underline{R}}^k \underline{\underline{R}}_0)^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}}^T \end{bmatrix} \left\{ - \begin{bmatrix} \underline{\underline{I}} & \widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix} \begin{Bmatrix} \Delta \underline{\dot{u}}^k \\ \Delta \underline{\omega}^k \end{Bmatrix} + \begin{Bmatrix} \Delta \underline{\dot{u}}^\ell \\ \Delta \underline{\omega}^\ell \end{Bmatrix} + \begin{bmatrix} \widetilde{\omega}^k & (\widetilde{\underline{\dot{u}} + \widetilde{u}\underline{\omega}^k}) \\ \underline{\underline{0}} & \underline{\underline{S}}^{-T} \underline{\underline{Z}}^k \end{bmatrix} \begin{Bmatrix} \Delta \underline{u}^k \\ \underline{\Delta\psi}^k \end{Bmatrix} + \begin{bmatrix} \widetilde{\omega}^{kT} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}}^{-T} \underline{\underline{Z}}^\ell \end{bmatrix} \begin{Bmatrix} \Delta \underline{u}^\ell \\ \underline{\Delta\psi}^\ell \end{Bmatrix} \right\},$$

where the following notation was used

$$\underline{\underline{G}}^k = \left[\underline{g}_1^k, \underline{g}_2^k, \underline{g}_3^k\right] = (\underline{\underline{R}}^k \underline{\underline{R}}_0)\frac{\widetilde{\omega}^\ell - \widetilde{\omega}^k}{2}, \quad \underline{\underline{G}}^\ell = \left[\underline{g}_1^\ell, \underline{g}_2^\ell, \underline{g}_3^\ell\right] = (\underline{\underline{R}}^\ell \underline{\underline{R}}_0)\frac{\widetilde{\omega}^\ell - \widetilde{\omega}^k}{2},$$

and

$$\underline{\underline{Z}}^k(\underline{g}^*) = \widetilde{g}_1^\ell \widetilde{b}_1^k + \widetilde{g}_2^\ell \widetilde{b}_2^k + \widetilde{g}_3^\ell \widetilde{b}_3^k, \quad \underline{\underline{Z}}^\ell(\underline{g}^*) = \widetilde{g}_1^k \widetilde{b}_1^\ell + \widetilde{g}_2^k \widetilde{b}_2^\ell + \widetilde{g}_3^k \widetilde{b}_3^\ell.$$

Combining the above results yields

$$\Delta \underline{F}^\ell = \Delta \begin{Bmatrix} \underline{h} \\ \underline{g} \end{Bmatrix} = \underline{\underline{G}}^{\ell k} \begin{Bmatrix} \Delta \underline{\dot{u}}^k \\ \Delta \underline{\omega}^k \end{Bmatrix} + \underline{\underline{G}}^{\ell\ell} \begin{Bmatrix} \Delta \underline{\dot{u}}^\ell \\ \Delta \underline{\omega}^\ell \end{Bmatrix} + \underline{\underline{K}}^{\ell k} \begin{Bmatrix} \Delta \underline{u}^k \\ \underline{\Delta\psi}^k \end{Bmatrix} + \underline{\underline{K}}^{\ell\ell} \begin{Bmatrix} \Delta \underline{u}^\ell \\ \underline{\Delta\psi}^\ell \end{Bmatrix},$$

where

$$\underline{\underline{G}}^{\ell k} = -\underline{\underline{Q}} \begin{bmatrix} \underline{\underline{I}} & \widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix}, \quad \underline{\underline{G}}^{\ell\ell} = \underline{\underline{Q}}.$$

and

$$\underline{\underline{K}}^{\ell k} = \begin{bmatrix} \underline{\underline{0}} & \widetilde{h}^T \\ \underline{\underline{0}} & \underline{\underline{Y}}^k \end{bmatrix} + \underline{\underline{Q}} \begin{bmatrix} \widetilde{\omega}^k & (\widetilde{\underline{\dot{u}} + \widetilde{u}\underline{\omega}^k}) \\ \underline{\underline{0}} & \underline{\underline{S}}^{-T} \underline{\underline{Z}}^k \end{bmatrix}, \quad \underline{\underline{K}}^{\ell\ell} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{Y}}^\ell \end{bmatrix} + \underline{\underline{Q}} \begin{bmatrix} \widetilde{\omega}^{kT} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}}^{-T} \underline{\underline{Z}}^\ell \end{bmatrix}.$$

The rotated damping matrix was defined as

$$\underline{\underline{Q}} = \begin{bmatrix} (\underline{\underline{R}}^k \underline{\underline{R}}_0) & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}} \end{bmatrix} \underline{\underline{Q}}^* \begin{bmatrix} (\underline{\underline{R}}^k \underline{\underline{R}}_0)^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}}^T \end{bmatrix}.$$

The linearization of the second half of the dissipative forces is

$$\Delta \underline{F}^k = \underline{\underline{G}}^{kk} \begin{Bmatrix} \Delta \underline{\dot{u}}^k \\ \Delta \underline{\omega}^k \end{Bmatrix} + \underline{\underline{G}}^{k\ell} \begin{Bmatrix} \Delta \underline{\dot{u}}^\ell \\ \Delta \underline{\omega}^\ell \end{Bmatrix} + \underline{\underline{K}}^{kk} \begin{Bmatrix} \Delta \underline{u}^k \\ \underline{\Delta\psi}^k \end{Bmatrix} + \underline{\underline{K}}^{k\ell} \begin{Bmatrix} \Delta \underline{u}^\ell \\ \underline{\Delta\psi}^\ell \end{Bmatrix},$$

where

$$\underline{\underline{G}}^{kk} = - \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{0}} \\ \widetilde{u} & \underline{\underline{I}} \end{bmatrix} \underline{\underline{G}}^{\ell k}, \quad \underline{\underline{G}}^{k\ell} = - \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{0}} \\ \widetilde{u} & \underline{\underline{I}} \end{bmatrix} \underline{\underline{G}}^{\ell\ell}.$$

and

$$\underline{\underline{K}}^{kk} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \widetilde{h}^T & \underline{\underline{0}} \end{bmatrix} - \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{0}} \\ \widetilde{u} & \underline{\underline{I}} \end{bmatrix} \underline{\underline{K}}^{\ell k}, \quad \underline{\underline{K}}^{k\ell} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} \\ \widetilde{h} & \underline{\underline{0}} \end{bmatrix} - \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{0}} \\ \widetilde{u} & \underline{\underline{I}} \end{bmatrix} \underline{\underline{K}}^{\ell\ell}.$$