Dymore User's Manual

Definition of beam properties

Contents

1 Construction of the stiffness matrix

The sectional properties of the beam are defined in the local axis system attached to the curve defining the geometry of the beam, as illustrated in fig. 1. Axis $\bar{e}_1$ is tangent to the curve, and axes $\bar{e}_2$ and $\bar{e}_3$ define the plane of the cross-section. The reference axis of the beam is the curve that defines the beam geometry.

1.1 Sign conventions

In this section, *all quantities are measured in this local coordinate system.* The 6×6 *sectional stiffness matrix,* $\underline{\underline{C}}$, relates the sectional axial strain, ϵ_1, transverse shearing strains, ϵ_2 and ϵ_3, twisting curvatures, κ_1 and two bending curvatures, κ_2 and κ_3, to the axial force, F_1, transverse shear forces, F_2 and F_3, twisting moment, M_1, and two bending moments, M_2 and M_3. The relationship between these sectional strains and sectional stress resultants takes the form of a symmetric, 6×6 matrix

$$\left\{\begin{matrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{matrix}\right\} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \left\{\begin{matrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{matrix}\right\}, \text{ or } \underline{F} = \underline{\underline{C}}\ \underline{\epsilon}. \tag{1}$$

The three forces F_1, F_2, and F_3 are *positive along axes* $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively, whereas moments M_1, M_2 and M_3 are *positive about axes* $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively, as depicted in fig. 2. Identical sign conventions are used for the three strains, ϵ_1, ϵ_2 and ϵ_3, and curvatures κ_1, κ_2 and κ_3, respectively. The three forces are the resultants of the stress distributions over the cross section; the three moments are computed with respect to the reference axis of the beam as depicted in fig. 2.

The 6×6 *sectional compliance matrix*, $\underline{\underline{S}}$, the inverse of the sectional stiffness matrix, *i.e.* $\underline{\underline{S}} = \underline{\underline{C}}^{-1}$, and hence,

$$\underline{\epsilon} = \underline{\underline{S}}\ \underline{F}. \tag{2}$$

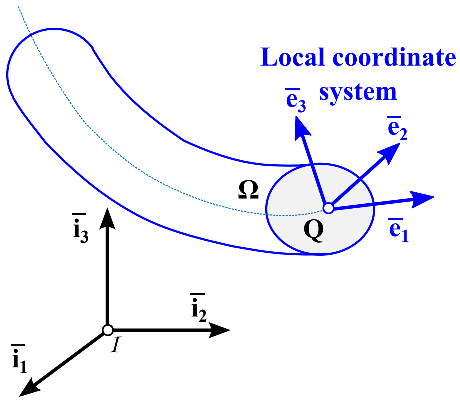

Figure 1: Configuration of the beam.

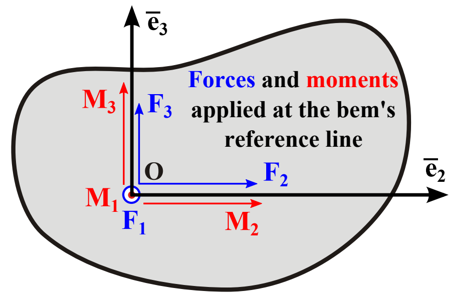

Figure 2: Sign conventions for the externally applied force and moment components acting on the cross-section.

1.2 Decomposition of the stiffness matrix

It is often the case that the 6×6 stiffness matrix defined by eq. (1) contains a number of vanishing terms and presents the following structure

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} C_{11} & 0 & 0 & 0 & C_{15} & C_{16} \\ 0 & C_{22} & C_{23} & C_{24} & 0 & 0 \\ 0 & C_{23} & C_{33} & C_{34} & 0 & 0 \\ 0 & C_{24} & C_{34} & C_{44} & 0 & 0 \\ C_{15} & 0 & 0 & 0 & C_{55} & C_{56} \\ C_{16} & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}. \tag{3}$$

In such cases, the complete problem splits into an *axial force-bending moment* problem characterized by the following 3×3 stiffness matrix

$$\begin{Bmatrix} F_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{15} & C_{16} \\ C_{15} & C_{55} & C_{56} \\ C_{16} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}, \tag{4}$$

and a *twisting moment-shear force* problem characterized by the following 3×3 stiffness matrix

$$\begin{Bmatrix} M_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} C_{44} & C_{24} & C_{34} \\ C_{24} & C_{22} & C_{23} \\ C_{34} & C_{23} & C_{33} \end{bmatrix} \begin{Bmatrix} \kappa_1 \\ \epsilon_2 \\ \epsilon_3 \end{Bmatrix}. \tag{5}$$

The corresponding compliance matrices are

$$\begin{Bmatrix} \epsilon_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{15} & S_{16} \\ S_{15} & S_{55} & S_{56} \\ S_{16} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} F_1 \\ M_2 \\ M_3 \end{Bmatrix}, \tag{6}$$

and

$$\begin{Bmatrix} \kappa_1 \\ \epsilon_2 \\ \epsilon_3 \end{Bmatrix} = \begin{bmatrix} S_{44} & S_{24} & S_{34} \\ S_{24} & S_{22} & S_{23} \\ S_{34} & S_{23} & S_{33} \end{bmatrix} \begin{Bmatrix} M_1 \\ F_2 \\ F_3 \end{Bmatrix}. \tag{7}$$

for the axial force-bending moment and twisting moment-shear force problems, respectively.

1.3 The axial force-bending moment problem

If the stiffness matrix of the cross-section presents the special structure displayed in eq. (3), it becomes possible to separately analyze the axial force-bending moment and twisting moment-shear force problems. The former problem is the focus of this section.

To further simplify the relationship between the axial forces and bending moment and the corresponding sectional strain components, eq. (4), it is convenient to introduce **the centroid of the cross-section**, a point of the cross-section with coordinates (x_{c2}, x_{c3}). With the help of the centroid, the relationship between axial force and axial strain decouples from the relationship between bending moments and curvatures,

$$F_1^c = S\ \varepsilon_1^c. \tag{8}$$

where $F_1^c = F_1$ is the axial force, ε_1^c the axial strain at the centroid and S the axial stiffness. The bending moments are related to the sectional curvatures,

$$\begin{Bmatrix} M_2^c \\ M_3^c \end{Bmatrix} = \begin{bmatrix} H_{22}^c & -H_{23}^c \\ -H_{23}^c & H_{33}^c \end{bmatrix} \begin{Bmatrix} \kappa_2^c \\ \kappa_3^c \end{Bmatrix} \tag{9}$$

where M_2^c and M_3^c are the bending moments computed with respect to the centroid about axes parallel to $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively, $\kappa_2^c = \kappa_2$ and $\kappa_3^c = \kappa_3$ the sectional curvatures, H_{22}^c and H_{33}^c the bending stiffnesses computed with respect to the centroid about axes parallel to $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively, and H_{23}^c the cross bending stiffness computed with respect to the centroid about axes parallel to $\bar{\imath}_2$ and $\bar{\imath}_3$.

The forces and moments computed with respect to the reference point and the centroid can be related as follows

$$\begin{Bmatrix} F_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x_{c3} & 1 & 0 \\ -x_{c2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_1^c \\ M_2^c \\ M_3^c \end{Bmatrix}; \qquad \begin{Bmatrix} F_1^c \\ M_2^c \\ M_3^c \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -x_{c3} & 1 & 0 \\ x_{c2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ M_2 \\ M_3 \end{Bmatrix}. \tag{10}$$

Similarly, the sectional strains and curvatures with respect to the reference point and the centroid can be related as follows

$$\begin{Bmatrix} \varepsilon_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_{c3} & -x_{c2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_1^c \\ \kappa_2^c \\ \kappa_3^c \end{Bmatrix}; \qquad \begin{Bmatrix} \varepsilon_1^c \\ \kappa_2^c \\ \kappa_3^c \end{Bmatrix} = \begin{bmatrix} 1 & -x_{c3} & x_{c2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}. \tag{11}$$

Eqs. (8) and (9) relating the sectional forces and strains about the centroid can be recast in a single matrix equation as

$$\begin{Bmatrix} F_1^c \\ M_2^c \\ M_3^c \end{Bmatrix} = \begin{bmatrix} S & 0 & 0 \\ 0 & H_{22}^c & -H_{23}^c \\ 0 & -H_{23}^c & H_{33}^c \end{bmatrix} \begin{Bmatrix} \varepsilon_1^c \\ \kappa_2^c \\ \kappa_3^c \end{Bmatrix}. \tag{12}$$

Introducing eqs. (10) and (11), the relationship between the corresponding quantities at the reference point are readily found as

$$\begin{Bmatrix} F_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} S & x_{c3}S & -x_{c2}S \\ x_{c3}S & H_{22}^c + x_{c3}^2 S & -(H_{23}^c + x_{c2}x_{c3}S) \\ -x_{c2}S & -(H_{23}^c + x_{c2}x_{c3}S) & H_{33}^c + x_{c2}^2 S \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}. \tag{13}$$

The corresponding bending compliance matrix is then found by inversion

$$\begin{Bmatrix} \varepsilon_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \frac{1}{\Delta_H} \begin{bmatrix} \frac{\Delta_H}{S} + x_{c2}^2 H_{22}^c + x_{c3}^2 H_{33}^c - 2x_{c2}x_{c3}H_{23}^c & x_{c2}H_{23}^c - x_{c3}H_{33}^c & x_{c2}H_{22}^c - x_{c3}H_{23}^c \\ x_{c2}H_{23}^c - x_{c3}H_{33}^c & H_{33}^c & H_{23}^c \\ x_{c2}H_{22}^c - x_{c3}H_{23}^c & H_{23}^c & H_{22}^c \end{bmatrix} \begin{Bmatrix} F_1 \\ M_2 \\ M_3 \end{Bmatrix}, \tag{14}$$

where $\Delta_H = H_{22}^c H_{33}^c - (H_{23}^c)^2$.

1.4 The twisting moment-shear force problem

If the stiffness matrix of the cross-section presents the special structure displayed in eq. (3), it becomes possible to separately analyze the axial force-bending moment and twisting moment shear force problems. The latter problem is the focus of this section.

To further simplify the relationship between the twisting moment and shearing forces and the corresponding sectional strain components, eq. (5), it is convenient to introduce **the shear center of the cross-section**, a point of the cross-section of coordinates (x_{k2}, x_{k3}). With the help of the shear center, the relationship between twisting moment and twist rate decouples from the relationship between shearing forces and sectional transverse strains,

$$M_1^k = J^k \ \kappa_1^k; \tag{15}$$

where M_1^k is the twisting moment computed with respect to the shear center, $\varepsilon_1^k = \varepsilon_1$ the sectional twist rate and J^k the torsional stiffness. The shear forces are related to the sectional transverse strains,

$$\begin{Bmatrix} F_2^k \\ F_3^k \end{Bmatrix} = \begin{bmatrix} K_{22}^k & -K_{23}^k \\ -K_{23}^k & K_{33}^k \end{bmatrix} \begin{Bmatrix} \gamma_{12}^k \\ \gamma_{13}^k \end{Bmatrix} \tag{16}$$

where $F_2^k = F_2$ and $F_3^k = F_3$ are the sectional shearing forces, γ_{12}^k and γ_{13}^k the sectional transverse shearing strains, K_{22}^k and K_{33}^k the shearing stiffnesses computed with respect to the shear center about axes parallel to $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively, and K_{23}^k the cross shearing stiffness computed with respect to the shear center about axes parallel to $\bar{\imath}_2$ and $\bar{\imath}_3$.

The forces and moments computed with respect to the reference point and the shear center can be related as follows

$$\begin{Bmatrix} M_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 1 & -x_{k3} & x_{k2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} M_1^k \\ F_2^k \\ F_3^k \end{Bmatrix}; \qquad \begin{Bmatrix} M_1^k \\ F_2^k \\ F_3^k \end{Bmatrix} = \begin{bmatrix} 1 & x_{k3} & -x_{k2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} M_1 \\ F_2 \\ F_3 \end{Bmatrix}. \tag{17}$$

Similarly, the sectional twist rate and transverse strains with respect to the reference point and shear center are related as follows

$$\begin{Bmatrix} \kappa_1 \\ \gamma_{12} \\ \gamma_{13} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x_{k3} & 1 & 0 \\ -x_{k2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \kappa_1^k \\ \gamma_{12}^k \\ \gamma_{13}^k \end{Bmatrix}; \qquad \begin{Bmatrix} \kappa_1^k \\ \gamma_{12}^k \\ \gamma_{13}^k \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -x_{k3} & 1 & 0 \\ x_{k2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \kappa_1 \\ \gamma_{12} \\ \gamma_{13} \end{Bmatrix}. \tag{18}$$

Eqs. (15) and (16) relating the sectional forces and strains about the shear center can be recast in a matrix form as

$$\begin{Bmatrix} M_1^k \\ F_2^k \\ F_3^k \end{Bmatrix} = \begin{bmatrix} J^k & 0 & 0 \\ 0 & K_{22}^k & -K_{23}^k \\ 0 & -K_{23}^k & K_{33}^k \end{bmatrix} \begin{Bmatrix} \kappa_1^k \\ \gamma_{12}^k \\ \gamma_{13}^k \end{Bmatrix}. \tag{19}$$

Introducing eqs. (17) and (18), the relationship between the corresponding quantities at the reference point are readily found as

$$\begin{Bmatrix} M_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} J + x_{k2}^2 K_{33}^k + x_{k3}^2 K_{22}^k + 2x_{k2}x_{k3}K_{23}^k & -x_{k2}K_{23}^k - x_{k3}K_{22}^k & x_{k2}K_{33}^k + x_{k3}K_{23}^k \\ -x_{k2}K_{23}^k - x_{k3}K_{22}^k & K_{22}^k & -K_{23}^k \\ x_{k2}K_{33}^k + x_{k3}K_{23}^k & -K_{23}^k & K_{33}^k \end{bmatrix} \begin{Bmatrix} \kappa_1 \\ \gamma_{12} \\ \gamma_{13} \end{Bmatrix} \tag{20}$$

The corresponding bending compliance matrix is then found by inversion

$$\begin{Bmatrix} \kappa_1 \\ \gamma_{12} \\ \gamma_{13} \end{Bmatrix} = \begin{bmatrix} 1/J & x_{k3}/J & -x_{k2}/J \\ x_{k3}/J & K_{33}^k/\Delta_K + x_{k3}^2/J & K_{23}^k/\Delta_K - x_{k2}x_{k3}/J \\ -x_{k2}/J & K_{23}^k/\Delta_K - x_{k2}x_{k3}/J & K_{22}^k/\Delta_K + x_{k2}^2/J \end{bmatrix} \begin{Bmatrix} M_1 \\ F_2 \\ F_3 \end{Bmatrix}, \tag{21}$$

where $\Delta_K = K_{22}^k K_{33}^k - (K_{23}^k)^2$.

2 Construction of the mass matrix

The 6×6 *sectional mass matrix*, M, relates the sectional linear velocities, denoted v_1, v_2 and v_3, and angular velocities, denoted ω_1, ω_2 and ω_3, to the sectional linear momenta, denoted p_1, p_2 and p_3, and angular momenta, denoted h_1, h_2 and h_3. The relationship between these sectional velocities and sectional momenta takes the form of a symmetric, 6×6 matrix

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ h_1 \\ h_2 \\ h_3 \end{Bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} & M_{36} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} & M_{46} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} & M_{56} \\ M_{16} & M_{26} & M_{36} & M_{46} & M_{56} & M_{66} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}. \tag{22}$$

Due to the nature of the problem, many of these coefficients vanish, and the remaining entries are written as

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ h_1 \\ h_2 \\ h_3 \end{Bmatrix} = \begin{bmatrix} m_{00} & 0 & 0 & 0 & m_{00}x_{3m} & -m_{00}x_{2m} \\ 0 & m_{00} & 0 & -m_{00}x_{3m} & 0 & 0 \\ 0 & 0 & m_{00} & m_{00}x_{2m} & 0 & 0 \\ 0 & -m_{00}x_{3m} & m_{00}x_{2m} & m_{11} & 0 & 0 \\ m_{00}x_{3m} & 0 & 0 & 0 & m_{22} & -m_{23} \\ -m_{00}x_{2m} & 0 & 0 & 0 & -m_{23} & m_{33} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}, \tag{23}$$

where m_{00} is the sectional mass per unit span and x_{2m} and x_{3m} the coordinates of the sectional center of mass. The sectional mass moments of inertia per unit span are defined as

$$m_{22} = \int_{\mathcal{A}} \rho x_3^2 \, \mathrm{d}\mathcal{A}, \tag{24a}$$

$$m_{33} = \int_{\mathcal{A}} \rho x_2^2 \, \mathrm{d}\mathcal{A}, \tag{24b}$$

$$m_{23} = \int_{\mathcal{A}} \rho x_2 x_3 \, \mathrm{d}\mathcal{A}, \tag{24c}$$

where m_{22} and m_{33} are the section mass moments of inertia per unit span with respect to unit vectors $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively, and m_{23} the sectional cross-product of inertia per unit span. The polar moment of inertia per unit span, denoted m_{11}, is given by

$$m_{11} = m_{22} + m_{33}. \tag{25}$$

2.1 Mass matrix evaluated by SectionBuilder

If the mass matrix is evaluated by SectionBuilder, all its entries are defined unequivocally and are available.

2.2 Mass matrix evaluated from sectional properties

It is often the case that sectional properties are estimated using strength of material approximations or experimental measurements. Consequently, not all properties might be available and several check are performed to ensure the consistency of the mass matrix.

- For instance, the moments of inertia m_{22} and m_{33} are often very small and measurements are seldom available. To deal with this case, the following procedure is followed. If $m_{22} = 0.0$ and $m_{33} = 0.0$, these moments of inertia will be selected as

$$m_{22} = \frac{m_{11}}{1 + (x_{m2}/x_{m3})^2}; \quad m_{33} = \frac{m_{11}}{1 + (x_{m3}/x_{m2})^2}. \tag{26}$$

  A warning is printed: "Mass moment of inertia about e2 or/and e3 axes are adjusted to ensure consistent mass matrix."

- The polar moment of inertia should be given by eq. (25). If $m_{11} \neq m_{22} + m_{33}$ these moments of inertia will be corrected as

$$m_{22\text{new}} = \frac{m_{11}}{1 + m_{33\text{old}}/m_{22\text{old}}}; \quad m_{33\text{new}} = \frac{m_{11}}{1 + m_{22\text{old}}/m_{33\text{old}}}. \tag{27}$$

  A warning is printed: "Mass moment of inertia about e2 or/and e3 axes are adjusted to ensure consistent mass matrix."

Because the cross-product of inertia m_{23} is rarely available, it is estimated as follows

$$m_{23} = m_{00} x_{m2} x_{m3}. \tag{28}$$

To ensure the positive-definiteness of the mass matrix, its eigenvalues, denoted μ_i, $i = 1, 2, \ldots, 6$, are computed. If $\mu_i < 0$ for any i, an error message if printed "Mass matrix is not positive-definite." Let $\mu_{\max}$ denote the maximum eigenvalue of the mass matrix; if $\mu_i < 0.01\mu_{\max}$ for any i, a warning is printed "Some eigenvalues of the mass matrix are small."

3 Damping coefficient

Damping in the beam can be modeled by viscous forces $\underline{F}_d$ proportional to the strain rates, $\underline{F}_d = \mu_s \, \underline{\underline{C}} \, \underline{\dot{e}}$, where μ_s is the damping coefficient, $\underline{e}$ the 6 strain components, and $\underline{\underline{C}}$ the cross-sectional stiffness matrix. $\underline{F}_d$, $\underline{e}$, and $\underline{\underline{C}}$ are all measured in a body attached coordinate system.