Dymore User's Manual
Laminated composite plates

Contents

As the use of composite materials becomes more widespread, *laminated composite plates* play an increasingly important role in many applications. These constructions consist of a number of lamina, often called "layers" or "plies," stacked on top of each other to form the plate, often called a "laminate."

In typical applications, each lamina is made by embedding in a matrix material fibers that are aligned in a single direction. A number of polymeric materials can be used as matrix materials. For advanced composite used in aerospace applications, thermoset materials such as epoxy have been used extensively as matrices. Different materials have been used for the fibers: glass, graphite, or boron have all found widespread application.

Plates will be described by the orientation angle characterizing each layer, starting from the bottom ply. For instance, notation $[\pm 45, 0_2, 0_2, \mp 45]$ describes an 8-ply laminate. The first layer, starting from the bottom of the plate, has a fiber angle of +45 degrees, the next a -45 degree orientation. Next come two lamina with a 0 degree angles, and so on up to the last layer at the top of the plate which has a +45 degree angle.

In many instances, plates possess mid-plane symmetry, *i.e.*, for each lamina below the mid-plane, there is a lamina of identical thickness, orientation angle, and position above the mid-plane. In such case, the top half of the plate is a mirror image of the bottom half. This symmetry is reflected in the notation as well. It is not necessary to repeat the description of the top half of the laminate: notation $[\pm 45, 0_2, 0_2, \mp 45]$ is equivalent to $[\pm 45, 0_2]_S$, where notation $[.]_S$ indicates the mid-plane symmetry.

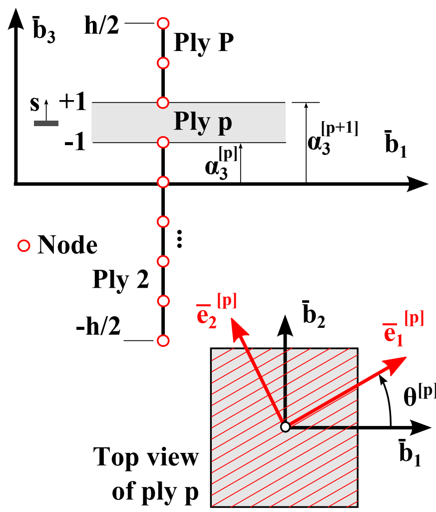

Figure 1: Description of the laminated composite plate.

1 Kinematics of the laminate problem

Figure 1 depicts a laminated composite plate with a thickness of h, consisting of P plies stacked on top of each other. Ply number 1 is at the bottom of the stacking sequence and ply number P is at the top. The plate's mid-plane is defined by base vectors $(\bar{b}_1, \bar{b}_2)$; base vector $\bar{b}_3$ is perpendicular to this mid-plane and defines the normal material line.

The bottom and top coordinates of ply number p are denoted $\alpha_3^{[p]}$ and $\alpha_3^{[p+1]}$, respectively. The thickness, $t^{[p]}$, of the ply is then

$$t^{[p]} = \alpha_3^{[p+1]} - \alpha_3^{[p]}. \tag{1}$$

To ease the manufacturing process, the various plies are often of the same thickness and made of the same material, although this is not necessary. Hybrid laminates, in which different materials are used for the various plies, are also found in specific applications.

To simplify the discretization process, a local non-dimensional variable s is defined within each ply. Within ply p, a linear transformation defines the change of variables from s to α_3 are related as follows

$$\alpha_3 = \frac{\alpha_3^{[p+1]} + \alpha_3^{[p]}}{2} + s\frac{t^{[p]}}{2}. \tag{2}$$

As illustrated in fig. 1, within ply p, s = -1 and +1 correspond to $\alpha_3 = \alpha_3^{[p]}$ and $\alpha_3^{[p+1]}$, respectively, *i.e.*, to the bottom and top edges of the ply, respectively. The Jacobian of the change of variables defined by eq. (2) is simply

$$\frac{\mathrm{d}\alpha_3}{\mathrm{d}s} = \frac{t^{[p]}}{2}. \tag{3}$$

Figure 1 also shows a top view of ply p, in which the fiber orientation angle becomes visible. In ply p, the material basis is denoted $\mathcal{E}^{+[p]} = (\bar{e}_1^{[p]}, \bar{e}_2^{[p]}, \bar{e}_3^{[p]})$; by convention, unit vector $\bar{e}_1^{[p]}$ is aligned with the fiber direction and the angle from base vector $\bar{b}_1$ to unit vector $\bar{e}_1^{[p]}$ is denoted $\theta^{[p]}$.

When resolved in the material basis, $\mathcal{E}^{+[p]}$, the material compliance matrix, eq. (6), of an orthotropic material takes the following form, $\underline{\epsilon}^+ = \underline{\underline{\mathcal{S}}}^+ \underline{\sigma}^+$, where notation $(\cdot)^+$ is used to denote tensor components resolved in the material basis, $\mathcal{E}^{+[p]}$; to simplify the notation, the superscript $(\cdot)^{[p]}$ was dropped from the sequel. The array of strain components resolved in the material basis is

$$\underline{\epsilon}^{+T} = \left\{ \epsilon_1^+, \epsilon_2^+, \epsilon_3^+, 2\epsilon_{23}^+, 2\epsilon_{13}^+, 2\epsilon_{12}^+ \right\}, \tag{4}$$

where ϵ_1^+, ϵ_2^+, and ϵ_3^+ are the axial strains components along unit vectors $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively, and $2\epsilon_{23}^+$, $2\epsilon_{13}^+$, and $2\epsilon_{12}^+$ are the engineering shear strains components in the planes normal to the same unit vectors, respectively. The array of stress components resolved in the material basis is

$$\underline{\tau}^{+T} = \left\{ \sigma_1^+, \sigma_2^+, \sigma_3^+, \sigma_{23}^+, \sigma_{13}^+, \sigma_{12}^+ \right\}, \tag{5}$$

where σ_1^+, σ_2^+, and σ_3^+ are the axial stress components along unit vectors $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively, and σ_{23}^+, σ_{13}^+ and σ_{12}^+ are the shear stresses acting in the planes normal to the same unit vectors, respectively.

Finally, for an orthotropic material, the components of the compliance matrix, $\underline{\underline{\mathcal{S}}}^+$, resolved in the material basis have the following form

$$\underline{\underline{\mathcal{S}}}^+ = \begin{bmatrix} 1/E_1^+ & -\nu_{21}^+/E_2^+ & -\nu_{31}^+/E_3^+ & 0 & 0 & 0 \\ -\nu_{12}^+/E_1^+ & 1/E_2^+ & -\nu_{32}^+/E_3^+ & 0 & 0 & 0 \\ -\nu_{13}^+/E_1^+ & -\nu_{23}^+/E_2^+ & 1/E_3^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12}^+ \end{bmatrix}. \tag{6}$$

The stiffness coefficients appearing in this compliance matrix are three distinct Young's moduli, E_1^+, E_2^+, and E_3^+, three Poisson's ratios, ν_{12}^+, ν_{13}^+, and ν_{23}^+, and three shearing moduli, G_{12}^+, G_{13}^+, and G_{23}^+. Because this matrix is symmetric, the following relationships hold

$$\nu_{23}^+/E_2^+ = \nu_{32}^+/E_3^+, \quad \nu_{31}^+/E_3^+ = \nu_{13}^+/E_1^+, \quad \nu_{21}^+/E_2^+ = \nu_{12}^+/E_1^+. \tag{7}$$

The stiffness form of the constitutive laws for an orthotropic material is obtained by inversion of eq. (6) to find

$$\underline{\sigma}^+ = \underline{\underline{\mathcal{C}}}^+ \underline{\epsilon}^+, \tag{8}$$

where $\underline{\underline{\mathcal{C}}}^+ = (\underline{\underline{\mathcal{S}}}^+)^{-1}$. Tedious algebra reveals that

$$\underline{\underline{\mathcal{C}}}^+ = \frac{1}{\alpha_0} \begin{bmatrix} E_1^+(1-\nu_{23}^+\nu_{32}^+) & E_1^+(\nu_{21}^+ + \nu_{31}^+\nu_{23}^+) & E_1^+(\nu_{31}^+ + \nu_{21}^+\nu_{32}^+) & 0 & 0 & 0 \\ E_2^+(\nu_{12}^+ + \nu_{32}^+\nu_{13}^+) & E_2^+(1-\nu_{13}^+\nu_{31}^+) & E_2^+(\nu_{32}^+ + \nu_{12}^+\nu_{31}^+) & 0 & 0 & 0 \\ E_3^+(\nu_{13}^+ + \nu_{23}^+\nu_{12}^+) & E_3^+(\nu_{23}^+ + \nu_{13}^+\nu_{21}^+) & E_3^+(1-\nu_{12}^+\nu_{21}^+) & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_0 G_{23}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_0 G_{13}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_0 G_{12}^+ \end{bmatrix}. \tag{9}$$

where

$$\alpha_0 = (1 - \nu_{23}^+\nu_{32}^+ - \nu_{13}^+\nu_{31}^+ - \nu_{12}^+\nu_{21}^+ - 2\nu_{32}^+\nu_{13}^+\nu_{21}^+). \tag{10}$$

Three material types are of practical importance: orthotropic, transversely isotropic, and isotropic materials. Their respective characteristics are as follows.

1. *Orthotropic materials* are characterized by nine independent constants: three Young's moduli, E_1^+, E_2^+, and E_3^+, three shear moduli, G_{12}^+, G_{13}^+, and G_{23}^+, and three Poisson's ratios, ν_{12}^+, ν_{13}^+, and ν_{23}^+.

2. *Transversely isotropic materials* are characterized by five independent constants: two Young's moduli, E_1^+ and E_2^+, one shear modulus, G_{12}^+, and two Poisson's ratios, ν_{12}^+ and ν_{23}^+. In this case, $E_3^+ = E_2^+$, $G_{13}^+ = G_{12}^+$ and $\nu_{13}^+ = \nu_{12}^+$. In view of the isotropy in plane $(\bar{e}_2, \bar{e}_3)$, subscripts $(\cdot)_2$ and $(\cdot)_3$ can be interchanged. Furthermore, the isotropy in plane $(\bar{e}_2, \bar{e}_3)$ implies $G_{23}^+ = E_2^+/[2(1+\nu_{23}^+)]$.

3. Finally, *isotropic material* are characterized by two independent constants: one Young's modulus, E, and one Poisson's ratio, ν. In this case, the isotropy of the material implies $E_1^+ = E_2^+ = E_3^+ = E$, $\nu_{12}^+ = \nu_{13}^+ = \nu_{23}^+ = \nu$, and $G_{12}^+ = G_{13}^+ = G_{23}^+ = E/[2(1+\nu)]$.

2 Rotation of material properties

Equations (8) define the material constitutive relationships resolved in the material basis, $\mathcal{E}^+$. The formulæ for the rotation of stress and strain components [1] can be written as $\underline{\underline{\sigma}}^* = \underline{\underline{R}}^T \underline{\underline{\sigma}}^+ \underline{\underline{R}}$ and $\underline{\underline{\varepsilon}}^* = \underline{\underline{R}}^T \underline{\underline{\varepsilon}}^+ \underline{\underline{R}}$, respectively, where $\underline{\underline{\sigma}}^*$ and $\underline{\underline{\sigma}}^+$ are the components of the stress tensor resolved in bases $\mathcal{B}^*$ and $\mathcal{M}^+$, respectively, and $\underline{\underline{\varepsilon}}^*$ and $\underline{\underline{\varepsilon}}^+$ are the components of the strain tensor resolved in the same bases, respectively. For the rotation illustrated in fig. 1, the rotation tensor is

$$\underline{\underline{R}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

where angle θ is measure algebraically from unit vector $\bar{b}_1$ to unit vector $\bar{e}_1$.

It will be convenient to recast the rotation formulæ in terms of the strain and stress arrays defined by eqs. (4) and (5), respectively, to find

$$\underline{\epsilon}^+ = \underline{\underline{R}}_\epsilon \underline{\epsilon}^*, \tag{12a}$$

$$\underline{\sigma}^* = \underline{\underline{R}}_\sigma \underline{\sigma}^+, \tag{12b}$$

where matrices $\underline{\underline{R}}_\epsilon$ and $\underline{\underline{R}}_\sigma$ are defined as

$$\underline{\underline{R}}_\epsilon = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & \cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & -\cos\theta\sin\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ -2\cos\theta\sin\theta & 2\cos\theta\sin\theta & 0 & 0 & 0 & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{13}$$

and

$$\underline{\underline{R}}_\sigma = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & -2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & 2\cos\theta\sin\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & 0 & 0 & 0 & \cos^2\theta - \sin^2\theta \end{bmatrix}, \tag{14}$$

respectively.

Introducing eqs. (12) into constitutive laws (8) leads to $\underline{\sigma}^* = (\underline{\underline{R}}_\sigma \underline{\underline{\mathcal{C}}}^+ \underline{\underline{R}}_\epsilon) \underline{\epsilon}^*$, revealing the components of the stiffness matrix resolved in basis $\mathcal{B}^*$ as

$$\underline{\underline{\mathcal{C}}}^* = \underline{\underline{R}}_\sigma \underline{\underline{\mathcal{C}}}^+ \underline{\underline{R}}_\epsilon. \tag{15}$$

Note that $\underline{\underline{R}}_\sigma = \underline{\underline{R}}_\epsilon^T$, and hence, the symmetry of the stiffness matrix in material basis $\mathcal{M}^+$ implies that of the same matrix in basis $\mathcal{B}^*$.

Given the components of the stiffness tensor resolved in the material basis, see eq. (9), the components of the same tensor resolved in basis $\mathcal{B}^*$ are of the following form

$$\underline{\underline{\mathcal{C}}}^* = \begin{bmatrix} C^*_{11} & C^*_{12} & C^*_{13} & 0 & 0 & C^*_{16} \\ C^*_{12} & C^*_{22} & C^*_{23} & 0 & 0 & C^*_{26} \\ C^*_{13} & C^*_{23} & C^*_{33} & 0 & 0 & C^*_{36} \\ 0 & 0 & 0 & C^*_{44} & C^*_{45} & 0 \\ 0 & 0 & 0 & C^*_{45} & C^*_{55} & 0 \\ C^*_{16} & C^*_{26} & C^*_{36} & 0 & 0 & C^*_{66} \end{bmatrix} \tag{16}$$

The tedious algebra expressed by eq. (15) then yields the desired stiffness components. For instance,

$$\begin{aligned} C^*_{11} = {} & \frac{C^+_{11} + 2C^+_{12} + C^+_{22}}{4} + \frac{C^+_{11} - 2C^+_{12} + C^+_{22}}{8} + \frac{C^+_{66}}{2} \\ & + \left(\frac{C^+_{11} - C^+_{22}}{2} \right) \cos 2\theta + \left(\frac{C^+_{11} - 2C^+_{12} + C^+_{22}}{8} - \frac{C^+_{66}}{2} \right) \cos 4\theta, \end{aligned} \tag{17}$$

where the powers of the trigonometric functions have been expressed in terms trigonometric functions of multiple of the angles using elementary trigonometric identities. Similar expressions can be found for all other components of the stiffness tensor.

2.1 Material invariants

To ease the evaluation of the stiffness components derived in the previous section, it is convenient to define the following sets of *material invariants*

$$\underline{\alpha}_I = \begin{Bmatrix} \alpha_{I1} \\ \alpha_{I2} \\ \alpha_{I3} \\ \alpha_{I4} \end{Bmatrix} = \frac{1}{8} \begin{Bmatrix} 2(C_{11}^+ + 2C_{12}^+ + C_{22}^+) \\ (C_{11}^+ - 2C_{12}^+ + C_{22}^+) + 4C_{66}^+ \\ 4(C_{11}^+ - C_{22}^+) \\ (C_{11}^+ - 2C_{12}^+ + C_{22}^+) - 4C_{66}^+ \end{Bmatrix}, \tag{18}$$

$$\underline{\alpha}_O = \begin{Bmatrix} \alpha_{O1} \\ \alpha_{O2} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} C_{13}^+ + C_{23}^+ \\ C_{13}^+ - C_{23}^+ \end{Bmatrix}, \tag{19}$$

and

$$\underline{\alpha}_S = \begin{Bmatrix} \alpha_{S1} \\ \alpha_{S2} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} C_{44}^+ + C_{55}^+ \\ C_{44}^+ - C_{55}^+ \end{Bmatrix}. \tag{20}$$

Next, the components of the stiffness matrix are partitioned in the following manner

$$\underline{C}_I^* = \begin{Bmatrix} C_{11}^* \\ C_{22}^* \\ C_{12}^* \\ C_{66}^* \\ C_{16}^* \\ C_{26}^* \end{Bmatrix}, \quad \underline{C}_O^* = \begin{Bmatrix} C_{13}^* \\ C_{23}^* \\ C_{36}^* \end{Bmatrix}, \quad \underline{C}_S^* = \begin{Bmatrix} C_{44}^* \\ C_{55}^* \\ C_{45}^* \end{Bmatrix}, \tag{21}$$

where arrays $\underline{C}_I^*$, $\underline{C}_O^*$, and $\underline{C}_S^*$ stores the in-plane, out-of-plane, and shear components, respectively. The stiffness components are now expressed in terms of the material invariants as

$$\underline{C}_I^* = \underline{\underline{\mathcal{X}}}_I \underline{\alpha}_I, \tag{22a}$$

$$\underline{C}_O^* = \underline{\underline{\mathcal{X}}}_O \underline{\alpha}_O, \tag{22b}$$

$$\underline{C}_S^* = \underline{\underline{\mathcal{X}}}_S \underline{\alpha}_S, \tag{22c}$$

where matrices $\underline{\underline{\mathcal{X}}}_I$, $\underline{\underline{\mathcal{X}}}_O$, and $\underline{\underline{\mathcal{X}}}_S$, functions of angle θ only, are defined as follows

$$\underline{\underline{\mathcal{X}}}_I(\theta) = \begin{bmatrix} 1 & 1 & \cos 2\theta & \cos 4\theta \\ 1 & 1 & -\cos 2\theta & \cos 4\theta \\ 1 & -1 & 0 & -\cos 4\theta \\ 0 & 1 & 0 & -\cos 4\theta \\ 0 & 0 & 1/2 \, \sin 2\theta & \sin 4\theta \\ 0 & 0 & 1/2 \, \sin 2\theta & -\sin 4\theta \end{bmatrix}, \tag{23}$$

$$\underline{\underline{\mathcal{X}}}_O(\theta) = \begin{bmatrix} 1 & \cos 2\theta \\ 1 & -\cos 2\theta \\ 0 & \sin 2\theta \end{bmatrix}, \tag{24}$$

and

$$\underline{\underline{\mathcal{X}}}_S(\theta) = \begin{bmatrix} 1 & \cos 2\theta \\ 1 & -\cos 2\theta \\ 0 & -\sin 2\theta \end{bmatrix}. \tag{25}$$

Of course, the last stiffness coefficient is unchanged by the rotation operation, *i.e.*, $C_{33}^* = C_{33}^+$.

References

[1] O.A. Bauchau and J.I. Craig. *Structural Analysis with Application to Aerospace Structures.* Springer, Dordrecht, Heidelberg, London, New-York, 2009.