Dymore User's Manual
Definition of mass properties

1 The parallel axis theorem

Figure 1 depicts the configuration of the rigid body: $\underline{s}$ is the position vector of a material point **Q** of the rigid body with respect to reference point **B**, and $\underline{q}$ is the position vector of the same point with respect to the center of mass **C**. Clearly, $\underline{s} = \underline{\eta} + \underline{q}$, where $\underline{\eta}$ is the position vector of the center of mass with respect to point **B**.

The tensor of mass moments of inertia evaluated with respect to point **B** is now

$$\underline{\underline{\varrho}}^B = \int_{\mathcal{V}} \widetilde{s}\,\widetilde{s}^T \ \rho \mathrm{d}\mathcal{V} = \int_{\mathcal{V}} (\widetilde{\eta} + \widetilde{q})(\widetilde{\eta}^T + \widetilde{q}^T) \ \rho \mathrm{d}\mathcal{V},$$

where $\mathcal{V}$ is the volume of the rigid body. Expanding the integrand and taking advantage of the fact that position vector $\underline{\eta}$ can be factored out of the integral sign leads to

$$\underline{\underline{\varrho}}^B = m \ \widetilde{\eta}\widetilde{\eta}^T + \widetilde{\eta} \left[\int_{\mathcal{V}} \widetilde{q}^T \ \rho \mathrm{d}\mathcal{V} \right] + \left[\int_{\mathcal{V}} \widetilde{q} \ \rho \mathrm{d}\mathcal{V} \right] \widetilde{\eta}^T + \int_{\mathcal{V}} \widetilde{q}\,\widetilde{q}^T \ \rho \mathrm{d}\mathcal{V}.$$

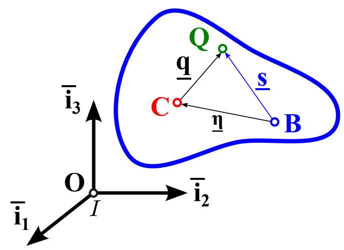

Figure 1: Evaluating the mass moments of inertia with respect to a reference point **B** and the center of mass **C**.

The two middle terms vanish because $\int_{\mathcal{V}} \widetilde{q} \ \rho \mathrm{d}\mathcal{V} = m\widetilde{r}_{CC} = 0$. The last term is the mass moment of inertia tensor, $\underline{\underline{\varrho}}^C$, evaluated with respect to the center of mass, and hence,

$$\underline{\underline{\varrho}}^B = \underline{\underline{\varrho}}^C + m \ \widetilde{\eta}\widetilde{\eta}^T. \tag{1}$$

Let ϱ^B_{ij} and ϱ^C_{ij} be the components of the mass moment of inertia tensors $\underline{\underline{\varrho}}^B$ and $\underline{\underline{\varrho}}^C$, respectively, and let $\left\{\eta_1, \eta_2, \eta_3\right\}^T$ be the components of vector $\underline{\eta}$, all resolved in the same basis. The diagonal components of tensor $\underline{\underline{\varrho}}^B$ now become

$$\varrho^B_{11} = \varrho^C_{11} + m(\eta_2^2 + \eta_3^2), \tag{2a}$$
$$\varrho^B_{22} = \varrho^C_{22} + m(\eta_1^2 + \eta_3^2), \tag{2b}$$
$$\varrho^B_{33} = \varrho^C_{33} + m(\eta_1^2 + \eta_2^2). \tag{2c}$$

Components ϱ^B_{11} and ϱ^C_{11} of the mass moment of inertia tensor are evaluated with respect to two different points, an arbitrary point **B** and the center of mass, respectively, but in the same basis, *i.e.*, with respect to parallel axis systems; hence, the name of *parallel axes theorem*.

The properties of the center of mass were used in the derivation of this theorem, hence, it is incorrect to write $\varrho^B_{11} = \varrho^R_{11} + m(\eta_2^2 + \eta_3^2)$ if points **B** and **R** are two arbitrary points of the rigid body.

Because the second term on the right-hand side of eqs. (2) is strictly positive, it follows $\varrho^B_{11} > \varrho^C_{11}$, that is, the moment of inertia always increases when moving away from the center of mass. In other words, the minimum value of ϱ_{11} is obtained when it is computed with respect to the center of mass.

The off-diagonal terms of tensor of moments of inertia are called *products of inertia*; in view of eq. (1), they become

$$\varrho^B_{23} = \varrho^C_{23} - m\eta_2\eta_3, \tag{3a}$$
$$\varrho^B_{13} = \varrho^C_{13} - m\eta_1\eta_3, \tag{3b}$$
$$\varrho^B_{12} = \varrho^C_{12} - m\eta_1\eta_2. \tag{3c}$$

In this case, the second term on the right-hand side could be positive or negative; consequently, products of inertia could increase of decrease when moving away from the center of mass.

2 Summary

Consider the components of the moments of inertia tensor resolved in **the local coordinate system**. Let the components of this tensor with respect to **a material point of the body** and with respect to the **center of mass of the body** be denoted

$$\underline{\underline{\varrho}}^B = \begin{bmatrix} \varrho_{11}^B & \varrho_{12}^B & \varrho_{13}^B \\ & \varrho_{22}^B & \varrho_{23}^B \\ & & \varrho_{33}^B \end{bmatrix} \quad \text{and} \quad \underline{\underline{\varrho}}^C = \begin{bmatrix} \varrho_{11}^C & \varrho_{12}^C & \varrho_{13}^C \\ & \varrho_{22}^C & \varrho_{23}^C \\ & & \varrho_{33}^C \end{bmatrix}, \tag{4}$$

respectively. The relationship between the components of these two tensors is readily found as

$$\underline{\underline{\varrho}}^B = \underline{\underline{\varrho}}^C + m \begin{bmatrix} \eta_2^2 + \eta_3^2 & -\eta_1\eta_2 & -\eta_1\eta_3 \\ -\eta_1\eta_2 & \eta_1^2 + \eta_3^2 & -\eta_2\eta_3 \\ -\eta_1\eta_3 & -\eta_2\eta_3 & \eta_1^2 + \eta_2^2 \end{bmatrix}, \tag{5}$$

where

$$\underline{\eta} = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix}. \tag{6}$$