The laminate shear modulus G can then be written as:

$$G = A_{66}/t = I_2 - R_2\varkappa \qquad \text{A5}$$

The laminate longitudinal elastic modulus can be found by assuming that there is no resultant hoop stress in the shaft, i.e., $N_{22} = 0$, so that:

$$N_{11} = (A_{11} - A_{12}^2/A_{22})\,\varepsilon_{11} \qquad \text{A6}$$

The laminate longitudinal elastic modulus E is found after some algebra to be:

$$E = \frac{4I_1(I_2 + R_2\varkappa) - \chi^2 R_1^2}{I_1 + I_2 - \chi R_1 + \varkappa R_2} \qquad \text{A7}$$

Our particular thin walled circular cross section will be defined by the following parameters:

r_m : midplane radius of the section,
t_o : thickness of one ply,
ϱ : density of the material.

Using our previous notation, we can now write:

$$\chi = (n_b \cos 2\theta_b + n \cos 2\theta)/(n + n_b),$$
$$\varkappa = (n_b \cos 4\theta_b + n \cos 4\theta)/(n + n_b). \qquad \text{A8}$$

The bending stiffness is now written as:

$$EI = \pi r_m^3\, t_o (n_b + n) E, \qquad \text{A9}$$

the shear stiffness as:

$$GK = \pi r_m\, t_o\, (n_b + n)\, G, \qquad \text{A10}$$

The transverse and rotary inertia are:

$$TI = 2\pi r_m\, t_o \varrho (n_b + n)$$
$$RI = \pi r_m^3\, t_o \varrho (n_b + n) \qquad \text{A11}$$

Optimal Design of High Speed Rotating Graphite/Epoxy Shafts

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ABSTRACT

An important problem in the design of a high speed rotating shaft of significant span is the lateral stability. With light weight Graphite/Epoxy material it is relatively easy to meet the torsional stiffness and strength requirements using a thin walled tube, but this might lead to a shaft which is sensitive to lateral buckling. This paper will address the following question: Is it possible to increase the first natural bending frequency of the shaft through an adequate tapering of the wall thickness? A numerical procedure using a generalized criterion of optimality is outlined. The shaft was modeled using a beam formulation including shear deformation and rotary inertia. The optimization was done under constant volume constraint. Different configurations are studied for various number of plies and orientation angle. In the optimal configuration the natural frequency increased by about 21 to 44%, and the bending stress decreased by about 48 to 59%.

CHARACTERIZATION OF THE SHAFT SECTION

WE WANT TO DEVELOP A METHOD THAT MAXIMIZES THE FIRST NATURAL bending frequency of a shaft. In any practical application this shaft would have to carry a required torque. The frequency optimization deals only with the bending stiffness of the shaft and does not take into account those torsional strength requirements. It is important to include those requirements within the optimization procedure otherwise we will produce meaningless results, i.e., shafts with a high natural bending frequency but with insufficient strength to carry the required torque.

This study deals with thin walled Graphite/Epoxy shafts of circular cross section. The torsional strength requirement can be translated into a wall thickness requirement. For this wall, the laminate configuration was chosen to consist of two parts:

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1) The "base" thickness is composed of n_b plies with a θ_b orientation angle. This base provides part of the torsional strength. Therefore, θ_b will be chosen to be 45° since this angle leads to the maximum torsional strength and stiffness [1].

2) The "variable" thickness consists of n plies with a θ orientation angle. Here n is a design variable to be calculated by the optimization procedure; θ is a parameter. We will optimize under the constraint that $n \geqslant n_{min}$, where n_{min} is a parameter which represents the minimum number of plies to be added to the base thickness. Thus, the torsional strength requirements will be met by $(n_b + n_{min})$ plies which are present all along the span of the shaft.

Now that we have chosen the laminate configuration, we can easily calculate the diffierent section properties of the shaft (bending and shear stiffness, transverse and rotary inertia) as a function of the design variable n and the different parameters n_b, θ, θ_b. (See Appendix)

DERIVATION OF THE OPTIMALITY CRITERION

Many different types of optimization algorithms have been used in structural problems. In the case of frequency optimization the optimality criterion approach has been shown to be efficient [2]. For this problem, we will derive a mathematical condition that will be satisfied if the spanwise distribution of the wall thickness of the shaft leads to its maximum natural bending frequency, under a constant mass constraint. To model the shaft, we will use a beam formulation including shear deformations and rotary inertia effects. We can start with the Rayleigh quotient expression for the natural frequency:

$$\omega_1^2 = \min \frac{\int_o^L EI(n) \left(\frac{d\alpha}{dx}\right)^2 dx + \int_o^L GK(n) \left(\frac{dw}{dx} - \alpha\right)^2 dx}{\int_o^L TI(n)\, w^2\, dx + \int_o^L RI(n)\, \alpha^2\, dx} \tag{1}$$

where

ω_1	is the first natural bending frequency,
x	is the coordinate along the span of the shaft of length L,
$EI(n)$	is the bending stiffness as a function of the design variable n,
$GK(n)$	is the shear stiffness,
$TI(n)$	is the transverse inertia,
$RI(n)$	is the rotary inertia,
$w(x)$	is the transverse deflection of the shaft,
$\alpha(x)$	is the rotation of an initially normal section
$\frac{d\alpha}{dx}$	is the curvature, and
$\frac{dw}{dx} - \alpha$	is the shearing angle.

Minimizing the Rayleigh quotient with respect to the functions $w(x)$ and $\alpha(x)$ leads to the standard eigenvalue problem. In order to maximize the frequency ω_1 with respect to the design variable $n(x)$ we must introduce the constant mass (or volume) constraint. So for a given initial mass of the tube, we seek to recorganize the thickness distribution for the maximum frequency. Using the Lagrangian multiplier technique, we can write a new functional:

$$(\omega_1^2)^* = \frac{\int_o^L EI(n) \left(\frac{d\alpha}{dx}\right)^2 dx + \int_o^L GK(n) \left(\frac{dw}{dx} - \alpha\right)^2 dx}{\int_o^L TI(n)\, w^2\, dx + \int_o^L RI(n)\, \alpha^2\, dx} - \lambda\left(\int_o^L v(n)\, dx - V\right) \tag{2}$$

where

λ	is a Langrangian multiplier,
$v(n)$	is the volume of the shaft per unit length of span,
V	is the total volume of the shaft.

It should be noted here that $n(x)$ is an integer function, since the number of plies has to be an integer. However, we will assume here $n(x)$ to be a continuous function. We will justify this assumption later on.

We can take the variation of this new functional with respect to $n(x)$. After some algebra we can find that the following relation must hold for every x where $n(x)$ is not constrained:

$$\left[\frac{\partial EI}{\partial n}\left(\frac{d\alpha}{dx}\right)^2 + \frac{\partial GK}{\partial n}\left(\frac{dw}{dx} - \alpha\right)^2\right] - \omega_1^2 \left[\frac{\partial TI}{\partial n} w^2 + \frac{\partial RI}{\partial n}\alpha^2\right] = \text{const.} \tag{3}$$

The right hand side constant can be found by multiplying both sides of equation (3) by $v(x)$ and integrating over the length L:

where

$$\text{const.} = U_s^*/V - U_k^*/V \tag{4}$$

$$U_s^* = \int_o^L \left[\frac{\partial EI}{\partial n}\left(\frac{d\alpha}{dx}\right)^2 + \frac{\partial GK}{\partial n}\left(\frac{dw}{dx} - \alpha\right)^2\right] v(x)\, dx,$$
$$U_k^* = \omega_1^2 \int_o^L \left[\frac{\partial TI}{\partial n} w^2 + \frac{\partial RI}{\partial n}\alpha^2\right] v(x)\, dx \tag{5}$$

so that U_s^*/V is the average gradient of strain energy density and U_k^*/V is the average gradient of kinetic energy density. Equation (3) can be rewritten as:

$$\left[\frac{\partial EI}{\partial n}\left(\frac{d\alpha}{dx}\right)^2 + \frac{\partial GK}{\partial n}\left(\frac{dw}{dx} - \alpha\right)^2\right] - \omega_1^2\left[\frac{\partial TI}{\partial n} w^2 + \frac{\partial RI}{\partial n}\alpha^2\right] = U_s^*/V - U_k^*/V \qquad (6)$$

This relation is the criterion for the maximum bending frequency. It can be read as follows: the first natural bending frequency of a shaft will be a maximum if the difference between the gradients of strain and kinetic energy densities is a constant along the span of the shaft and equals the difference of the averages of the same quantities.

In the frame of the finite element method, we can easily calculate the gradient of strain and kinetic energy by summing up the contribution of each element. If numerical integration is used, the procedure reduces to the summation of the contribution for each Gaussian point. The continuous function $n(x)$ is replaced by discrete values $n_i = n(x_i)$. (The subscript i means that we refer to the i^{th} Gaussian point.) Within this mathematical formalism, we can evaluate expressions (5) as follows:

$$\begin{aligned} U_s^* &= \sum_i \left\{\left[\frac{\partial EI}{\partial n}\left(\frac{d\alpha}{dx}\right)^2 + \frac{\partial GK}{\partial n}\left(\frac{dw}{dx} - \alpha\right)^2\right] v(x)\right\}_{x=x_i} w_i \\ &= \sum_i U_{si}^* \\ U_k^* &= \omega_1^2 \sum_i \left\{\left[\frac{\partial TI}{\partial n} w^2 + \frac{\partial RI}{\partial n}\alpha^2\right] v(x)\right\}_{x=x_i} \overline{w}_i \\ &= \sum_i U_{ki}^* \end{aligned} \qquad (7)$$

where w_i is the weight associated with the i^{th} Gaussian point.

We have derived the optimality criterion (3) in the continuous case, but in the finite element approach we will satisfy this condition only in an average sense, i.e., only at each Gaussian point. We can multiply both sides of equation (3) by $v(x)$ and integrate on a small volume located at Gaussian point i:

$$U_{si}^*/V_i - U_{ki}^*/V_i = \text{const.} \qquad (8)$$

where

$$V_i \text{ is defined by } V = \sum_i v(x_i)\,\overline{w}_i = \sum_i V_i$$

As we mentioned earlier, the optimality criterion (3) only holds where n is not constrained. This means that equation (8) only holds at the Gaussian points where n_i is not prescribed. To find the constant in equation (8), we can sum over the unconstrained Gaussian points; this operation is noted $\sum_{iu}$:

$$U_{si}^*/V_i - U_{ki}^*/V_i = U_{su}^*/V_u - U_{ku}^*/V_u = \varepsilon \qquad (9)$$

where

$$\begin{aligned} U_{su}^* &= \sum_{iu} U_{si}^* \\ U_{ku}^* &= \sum_{iu} U_{ku}^* \\ V_u &= \sum_{iu} V_i \end{aligned}$$

It is convenient to define here the following ratio:

$$\begin{aligned} R_i &= \frac{U_{si}^*/V_i}{U_{ki}^*/V_i + \varepsilon} && \text{if } \varepsilon > 0 \\ &= \frac{U_{si}^*/V_i + |\varepsilon|}{U_{ki}^*/V_i} && \text{if } \varepsilon < 0 \end{aligned} \qquad (10)$$

In the finite element frame, the optimality criterion reads as follows: the first natural bending fequency of a shaft will be a maximum when the ratios R_i are equal to unity at each Gaussian point where the design variable is not constrained. We can now use the optimality criterion to derive a recurrence relation. Starting with a constant thickness shaft we can use the finite element procedure to calculate eigen frequencies, then the gradients of strain and kinetic energies and finally the ratios R_i. We can use these ratios to compute a better approximation to the optimal thickness distribution. Then by successive iterations, we can obtain the optimal distribution.

If we have reached iteration ξ, we will use the following relation to calculate the (ξ +1) approximation to the thickness distribution:

$$n_i^{\xi+1} = C^{\xi+1}(n_b + n_i^{\xi})\,(R_i^{\xi})^P - n_b \qquad (11)$$

where superscripts indicate iteration number, and

$C^{\xi+1}$ is a normalizing factor,
P is a numerical damping factor.

The normalizing factor $C^{\xi+1}$ is obtained by the constant volume constraint:

$$C^{\xi+1} = V_u / \sum_{iu} V_i\,(R_i^{\xi})^P \qquad (12)$$

In this study, a value of $p = .25$ was found to be suitable. To test the convergence of the procedure, a norm was computed at each iteration. It is defined as

$$\text{norm} = \sum_{i\,u} [R_i^{\xi} - 1]^2 \tag{13}$$

When norm $<10^{-3}$ convergence was considered to be reached.

Relation [3] which can be used in an iterative optimization algorithm: starting from a constant thickness shaft, the optimal thickness distribution is reached by successive approximation.

NUMERICAL RESULTS

The optimization algorithm was applied to a Graphite/Epoxy cylindrical drive shaft with the following geometry: a span of 1.5 m, a mid plane radius of the cross section of 44×10^{-3} m and a ply thickness of $.144 \times 10^{-3}$ m. The material (HERCULES AS1/3501-6) has the following stiffness properties: $E_L = 130$ GPa, $E_T = 10$ GPa, $\nu_{LT} = .28$, $G_{LT} = 6$ GPa. The density is 1500 kg/m^3.

In the finite element analysis, a 4 noded beam element was used with a cubic interpolation for both lateral deflection and rotation. A 4 point Gaussian integration scheme was used so that there were 4 design variables n_i per element. Due to the symmetry of the problem, only half of the span of the shaft was molded using 4 elements. The shaft is assumed clamped at the root ($w = 0$, $\alpha = 0$) and the symmetry condition ($\alpha = 0$) is applied at mid-span.

The optimization algorithm was used to obtain the optimal thickness distribution of shafts with different values of the main parameters. The procedure was started from a constant thickness shaft with n_{init} plies. Two families of shafts were analysed for $n_{\text{init}} = 10$ and 12. For the first family ($n_{\text{init}} = 10$) the base thickness n_b was chosen to be 4 and $\theta_b = 45°$. During the optimization procedure, the design variable was constrained to a minimum $n_{\min} = 2$ then 4. In each case, three different angles were chosen $\theta = 0°$, $15°$ and $30°$. For the second family ($n_{\text{init}} = 12$), the corresponding parameters were $n_b = 6$, $\theta_b = 45$, $n_{\min} = 0$ then 2, and $\theta = 0°$, $15°$ and $30°$.

The thickness distribution profiles obtained from the code are plotted in Figures 1 and 2 for certain values of the different parameters. A first way of analyzing the results of the optimization is to look at the frequency increase for the different cases. Table 1 summarizes the bending frequencies in the original configuration f_c (constant thickness distribution), the bending frequencies in the optimized configuration f_o and the percentage change between those two frequencies $\Delta = 100\ (f_o - f_c)/f_c\%$. A significant increase is obtained in each case. A graphic representation of these results is given in Figure 3.

In order to make use of those results for the manufacturing of an actual shaft, the theoretical profile must be somewhat modified. First we can only use an integer number of plies; we must also avoid abrupt changes in

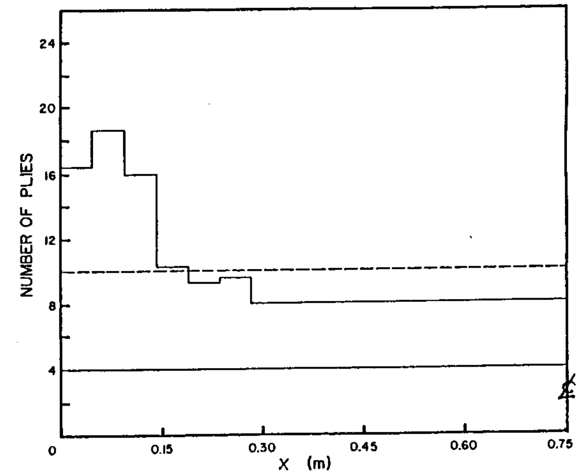

Figure 1. *Optimized thickness distribution, 1st family ($n_{\text{init}} = 10$, $n_{\min} = 4$, $n_b = 4$, $\theta_b = 45°$, $\theta = 15°$).*

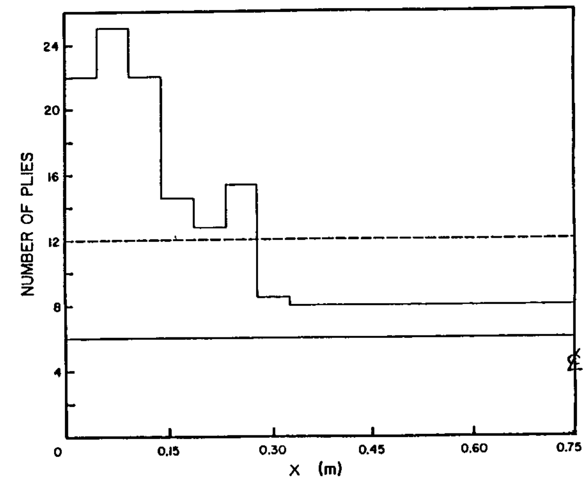

Figure 2. *Optimized thickness distribution, 2nd family ($n_{\text{init}} = 12$, $n_{\min} = 2$, $n_b = 6$, $\theta_b = 45°$, $\theta = 15°$).*

thickness that would result in stress concentration areas. These considerations and a desire for simplicity has led to the design shown in Figure 4 for $n_{\text{init}} = 10$. Since the theoretical profiles are only slightly modified by the different values of the angle θ, the same actual design was used for all the values of θ in the subsequent stress analysis.

In the derivation of the equations, we mentioned that it is an assumption to consider $n(x)$ to be a continuous function. We can now justify the assumption

Table 1. Effect of Optimization on Frequencies

a) 1st Family, $n_{init} = 10$, $n_b = 4$, $\theta_b = 45°$ Frequencies in (Hz)

	$\theta = 0°$	$\theta = 15°$	$\theta = 30°$
$n_{min} = 2$	$f_c = 340$, $f_o = 463$ $\Delta = 36\%$	$f_c = 320$, $f_o = 438$ $\Delta = 37\%$	$f_c = 248$, $f_o = 344$ $\Delta = 38\%$
$n_{min} = 4$	$f_c = 340$, $f_o = 410$ $\Delta = 21\%$	$f_c = 320$, $f_o = 388$ $\Delta = 21\%$	$f_c = 248$, $f_o = 301$ $\Delta = 21\%$

b) 2nd Family, $n_{init} = 12$, $n_b = 6$, $\theta_b = 45°$

	$\theta = 0°$	$\theta = 15°$	$\theta = 30°$
$n_{min} = 0$	$f_c = 325$, $f_o = 445$ $\Delta = 37\%$	$f_c = 304$, $f_o = 421$ $\Delta = 39\%$	$f_c = 239$, $f_o = 344$ $\Delta = 44\%$
$n_{min} = 2$	$f_c = 325$, $f_o = 423$ $\Delta = 30°$	$f_c = 304$, $f_o = 399$ $\Delta = 31\%$	$f_c = 239$, $f_o = 317$ $\Delta = 32\%$

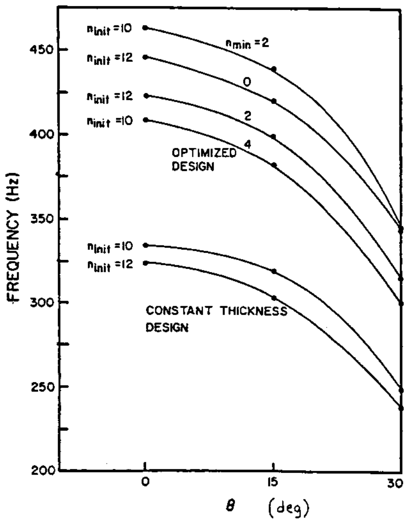

Figure 3. Effect of optimization on frequencies.

by comparing Figures 1 and 4. Figure 1 is the calculated optimal profile ($n(x)$ is not an integer), and Figure 4 is an approximate optimal profile ($n(x)$ is rounded to an integer). Still, the difference in frequencies for the two profiles is very little (less than 0.1%). In fact, around the optimal solution, the optimal frequency is insensitive to small changes in thickness, and rounding $n(x)$ to an integer has little effect. Taking $n(x)$ to be a continuous function is thus a reasonable assumption.

Another way of analyzing the effects of optimization is to look at the bending stress distribution along the shaft. To compare the different designs, it

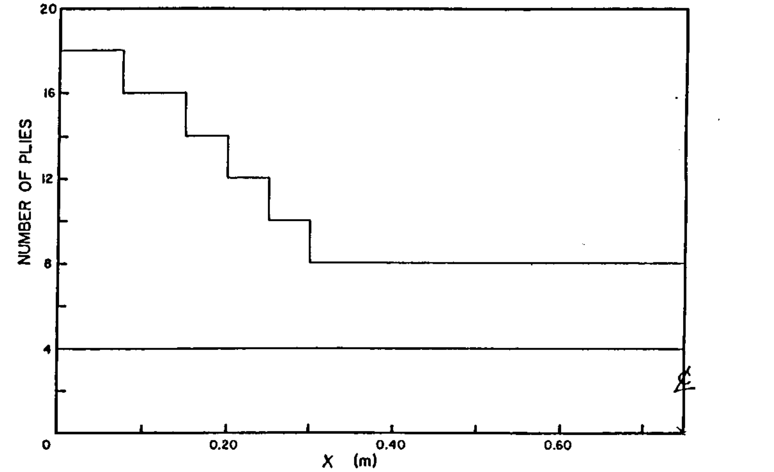

Figure 4. Actual design for an optimal shaft ($n_{init} = 10$, $n_{min} = 4$, $n_b = 4$, $\theta_b = 45°$, $\theta = 15°$).

was assumed in each case that the shaft had a standard initial imperfection, chosen to be a sinewave of .01 m maximum amplitude at midspan. The rotation frequency was 100 Hz. The same finite element model as previously described was used in a modal analysis to calculate the stress distribution. Table 2 gives the bending stress at the root of the shaft in the original configuration σ_o (constant thickness distribution), in the optimized configuration σ_o) and the percentage decrease between those stresses $\Delta = 100 (\sigma_o - \sigma_c)/\sigma_c\%$. The investigation was narrowed down to the actual designs that provide a sufficient torsional strength, i.e., $n_b = 4$, $n_{min} = 4$ for the first family and $n_b = 6$, $n_{min} = 2$ for the second family [1]. A graphic representation of these results is given in Figure 5.

The above results show that the optimization increases the natural bending frequency by about 21% to 31% and decreases the stress at the root of the shaft by about 48% to 59%. The parameters n_b, θ_b and n_{min} have been chosen to meet the torsional strength requirements but we are still free to choose θ. Previous research [4] has shown that 0° plies decrease drastically the ultimate torsional strength of the shaft because of a delamination behavior. To avoid this phenomenon, an angle $\theta = 15°$ is preferable rather than $\theta = 30°$ that causes a significant natural frequency drop.

Table 2. Effect of Optimization on Bending Stresses. Stresses in [MPa].

	$\theta = 0°$	$\theta = 15°$	$\theta = 30°$
$n_{init} = 12$ $n_{min} = 4$	$\sigma_c = 40.8$, $\sigma_o = 21.4$ $\Delta = -48\%$	$\sigma_c = 41.3$, $\sigma_o = 21.5$ $\Delta = -48\%$	$\sigma_c = 44.3$, $\sigma_o = 22.0$ $\Delta = -50\%$
$n_{init} = 12$ $n_{min} = 2$	$\sigma_c = 41.1$, $\sigma_o = 18.2$ $\Delta = -56\%$	$\sigma_c = 41.7$, $\sigma_o = 18.1$ $\Delta = -57\%$	$\sigma_c = 44.9$, $\sigma_o = 18.3$ $\Delta = -59\%$

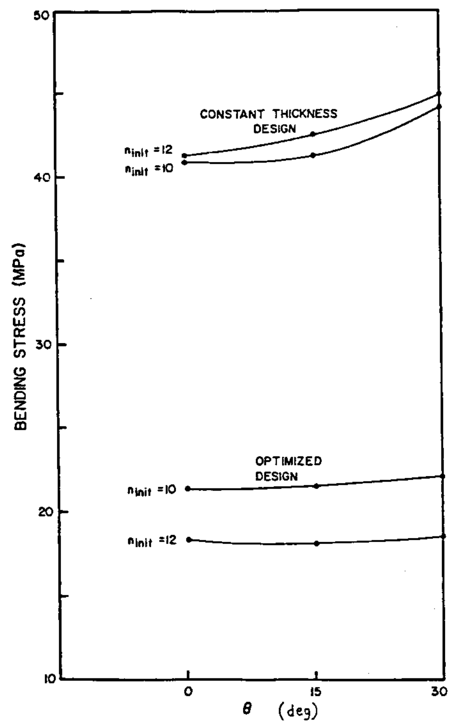

***Figure 5.** Effect of optimization on bending stress.*

The final design selected for manufacture of an actual shaft had the following values for the different parameters: $n_{\text{init}} = 10$, $n_b = 4$, $\theta_b = 45°$, $\theta = 15°$ and $n_{\min} = 4$. In the fabrication process, a different stacking sequence of the plies was used: the discontinuous plies were placed in between the continuous plies to alleviate the stress concentration at the discontinuities.

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REFERENCES

1. Bauchau, O. A., "Measurements of the Ultimate Torsional Strength of Graphite/Epoxy Tubes," To appear in *J. of Composite Materials.*
2. Kamat, M. P. and Simitses, G. J., "Optimal Beam Frequencies by the Finite Element Displacement Method," *Int. J. Solids and Structures,* Vol. 9, pp. 415–429, (1973).

3. Simitses, G. J., Kamat, M. P. and Smith, C. V., "The Strongest Column by the Finite Element Displacement Method," *AIAA Papers* Nos. 72-141 (1972). Also *AIAA J.,* (Synoptic), Vol. 11, No. 9.
4. Bauchau, O. A., "Design, Manufacturing, and Testing of High Speed Rotating Graphite/Epoxy Shafts," MIT, Department of Aeronautics and Astronautics, Ph.D. Thesis, 1981.
5. Tsai, S. W. and Hahn, T. H., "Introduction to Composite Materials," Technomic Publishing Co., 1980.

APPENDIX

In order to calculate the section properties, we have to relate the laminate elastic moduli to the basic elastic constants of the material. It can be seen that in our problem one of the principal axes of the laminate is parallel to the axis of symmetry of the tube and hence there is no coupling between inplane shearing and inplane extension. The inplane stress resultants per unit length N_{11}, N_{22}, N_{12} are related to the laminate inplane strains ε_{11}, ε_{22}, γ_{12} by

$$\begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} \qquad \text{A1}$$

According to classical lamination theory, the stiffness coefficients A_{rs} of the laminate are found by summing up the contributions of the different plies

$$A_{rs} = \sum_{i=1}^{N} t_i\, Q^i_{rs} \qquad \text{A2}$$

where t_i is the thickness of the i^{th} ply; Q^i_{rs} are the elastic moduli of the i^{th} ply, in laminate axis; and N is the total number of plies.

It is convenient to relate the elastic module Q^i_{rs} to the invariants of the material I_1, I_2, R_1, R_2 [5].

$$\begin{aligned} Q^i_{11} &= I_1 + I_2 + R_1 \cos 2\theta_1 + R_2 \cos 4\theta_i \\ Q^i_{22} &= I_1 + I_2 - R_1 \cos 2\theta_i + R_2 \cos 4\theta_i \\ Q^i_{12} &= I_1 - I_2 - R_2 \cos 4\theta_i \\ Q^i_{66} &= I_2 - R_2 \cos 4\theta_i \end{aligned} \qquad \text{A3}$$

If t is the total thickness of the laminate, we can define:

$$\begin{aligned} \chi &= \sum_{i=1}^{N} \frac{t_i}{t} \cos 2\theta_i \\ \kappa &= \sum_{i=1}^{N} \frac{t_i}{t} \cos 4\theta_i \end{aligned} \qquad \text{A4}$$