

A SOLUTION OF THE EIGENPROBLEM FOR UNDAMPED GYROSCOPIC SYSTEMS WITH THE LANCZOS ALGORITHM

O. A. BAUCHAU

*Rensselaer Polytechnic Institute, Department of Mechanical Engineering, Aeronautical Engineering and Mechanics,
Troy, New York 12180-3590, U.S.A.*

SUMMARY

This paper presents an efficient numerical solution of the quadratic eigenproblem arising in the analysis of gyroscopic systems. Such problems are known to reduce to a generalized linear eigenproblem defined by two real non-singular matrices, one symmetric and one skew-symmetric. For this class of problems, the general Lanczos algorithm for unsymmetric matrices is shown to simplify considerably and yields an efficient solution of the problem. Full advantage can be taken of the sparsity of the matrices and of the specific nature of gyroscopic systems. Numerical examples are presented, which demonstrate the efficiency and accuracy of the solution procedure.

1. INTRODUCTION

The equations of motion arising in the dynamic analysis of structures modelled with the finite element method¹ can be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}_L\mathbf{q} = \mathbf{f}(t) \quad (1)$$

where \mathbf{K}_L is the linear stiffness matrix, \mathbf{M} the mass matrix, $\mathbf{f}(t)$ the time dependent external force vector, and \mathbf{q} the nodal displacements vector. The matrices \mathbf{K}_L and \mathbf{M} are symmetric, positive definite and sparse. For complex structures, the order n of these matrices is large, typically a few thousands. The natural vibration frequencies ω_i and the corresponding mode shapes \mathbf{q}_i of system (1) are of prime interest to the designer, and are solutions of the following eigenproblem:

$$(\mathbf{K}_L - \omega_i^2\mathbf{M})\mathbf{q}_i = \mathbf{0} \quad (2)$$

The subspace iteration method^{1,2} developed originally [Reference 3, p. 337] by Clint and Jennings⁴ as the simultaneous iteration method, is now well established as the standard tool for solving such large eigenproblems, but, more recently, the Lanczos method^{5,6} has been gaining increasing popularity for the solution of this problem.

When dealing with a spinning structure it is often convenient to work with a body-attached coordinate system to calculate the stiffness matrix, but the inertia forces are more involved, leading to the following equations of motion:^{7,8}

$$({}^c\mathbf{K}_L + {}^c\mathbf{K}_{NL} - {}^c\mathbf{K}_C)\mathbf{q} - {}^c\mathbf{G}\dot{\mathbf{q}} + {}^c\mathbf{M}\ddot{\mathbf{q}} = \mathbf{f}(t) \quad (3)$$

where ${}^c\mathbf{K}_C$ is the centripetal acceleration matrix, ${}^c\mathbf{G}$ the gyroscopic matrix (due to Coriolis accelerations) and ${}^c\mathbf{K}_{NL}$ contains the modifications of the structural stiffness due to spin-induced

loads (the so-called non-linear stiffness matrix). All these matrices are symmetric and positive definite of order n , except for the gyroscopic matrix which is skew-symmetric. The left superscript 'e' indicates that all the matrices must be calculated about the steady state equilibrium position of the spinning body. Note that this steady state position may involve large displacements from the unstressed position of the structure.

System (3) appears in the analysis of numerous problems, for instance helicopter rotor blades⁹ or spin stabilized satellites with elastic appendages such as solar panels or antennas.^{7,8} Here again, the natural vibration frequencies λ_i and the corresponding mode shapes \mathbf{q}_i are important to the designer, and are solutions of the following quadratic eigenproblem:

$$({}^e\mathbf{K} - \lambda_i {}^e\mathbf{G} + \lambda_i^2 {}^e\mathbf{M})\mathbf{q}_i = \mathbf{0} \quad (4)$$

where ${}^e\mathbf{K} = {}^e\mathbf{K}_L + {}^e\mathbf{K}_{NL} - {}^e\mathbf{K}_C$. It is well known¹⁰ that this quadratic eigenproblem can be reduced to a linear eigenproblem by doubling the order of the system:

$$\begin{bmatrix} ({}^e\mathbf{M}) & \mathbf{0} \\ \mathbf{0} & ({}^e\mathbf{K}) \end{bmatrix} \begin{bmatrix} \lambda_i \mathbf{q}_i \\ \mathbf{q}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} & ({}^e\mathbf{M}) \\ -({}^e\mathbf{M}) & ({}^e\mathbf{G}) \end{bmatrix} \begin{bmatrix} \lambda_i \mathbf{q}_i \\ \mathbf{q}_i \end{bmatrix} \quad (5)$$

or

$$\mathbf{K}\mathbf{y}_i = \lambda_i \mathbf{M}\mathbf{y}_i \quad (6)$$

\mathbf{K} is now a symmetric, positive definite matrix and \mathbf{M} a skew-symmetric matrix, both of order $2n$.

Gupta^{11,12} has proposed a solution procedure for system (6) based on a combined Sturm sequence and inverse iteration technique. In this procedure, the calculation of each eigenvalue requires several factorizations of the matrix $\mathbf{K} - \lambda\mathbf{M}$, for various trial values of λ , so that it becomes less attractive for systems with a large bandwidth.¹

An alternative solution procedure for system (6) was described by Meirovitch.¹³ This procedure takes advantage of the specific nature of gyroscopic problems (namely the fact that one matrix is symmetric and the other skew-symmetric) to reduce (6) to a symmetric eigenvalue problem. Unfortunately, this procedure destroys the sparsity of the system, limiting its applicability to small size problems.

On the other hand, the subspace iteration method^{2,14} seems to be well suited for the solution of (6). However, since \mathbf{M} is skew-symmetric, the eigenvalues λ_i as well as the eigenvectors \mathbf{y}_i are complex quantities, so that complex arithmetic must be adopted right from the beginning of the iteration process. This considerably increases the storage and arithmetic cost, and the specific nature of the problem cannot be taken advantage of. Two recent survey papers^{15,16} provide the relative efficiencies of existing eigenproblem solution techniques pertaining to spinning structures, among others.

Recent publications^{5,6} have demonstrated the power of the Lanczos method applied to symmetric eigenvalue problems. It is thus natural to consider the application of the Lanczos algorithm to gyroscopic systems. Since one of the matrices involved is unsymmetric, Section 2 will describe a version of the Lanczos algorithm for general unsymmetric matrices, and Section 3 will take advantage of the specific nature of gyroscopic systems to considerably simplify the general algorithm. Finally, Section 5 presents some numerical results.

2. THE LANCZOS ALGORITHM FOR UNSYMMETRIC MATRICES

Several versions of the general Lanczos algorithm have been described in the literature.^{6,17} A version suitable for the solution of (6) is presented here. First we rewrite system (6) as

$$\mathbf{M}\mathbf{y} = \mu\mathbf{K}\mathbf{y} \quad (7)$$

where $\mu = 1/\lambda$. We also consider the adjoint problem

$$\bar{\mathbf{y}}^T \mathbf{M} = \mu \bar{\mathbf{y}}^T \mathbf{K} \quad (8)$$

\mathbf{y} and $\bar{\mathbf{y}}$ are the right and left eigenvectors, respectively. The subscript i appearing in (6) has been dropped here for clarity, and an overbar will be used on all quantities pertaining to the left eigenvectors. We now define two general linear transformations:

$$\mathbf{y} = \mathbf{Q}\mathbf{s} \quad \text{and} \quad \bar{\mathbf{y}} = \bar{\mathbf{Q}}\bar{\mathbf{s}} \quad (9)$$

where \mathbf{Q} and $\bar{\mathbf{Q}}$ are arbitrary rectangular matrices. A straightforward application of the Rayleigh minimum principle [Reference 1, p. 586] yields the following reduced eigenproblems:

$$(\bar{\mathbf{Q}}^T \mathbf{M} \mathbf{Q})\mathbf{s} = \mu (\bar{\mathbf{Q}}^T \mathbf{K} \mathbf{Q})\mathbf{s} \quad \text{and} \quad \bar{\mathbf{s}}^T (\bar{\mathbf{Q}}^T \mathbf{M} \mathbf{Q}) = \mu \bar{\mathbf{s}}^T (\bar{\mathbf{Q}}^T \mathbf{K} \mathbf{Q}) \quad (10)$$

In the Lanczos algorithm, the matrices \mathbf{Q} and $\bar{\mathbf{Q}}$ are constructed by means of the following three-term recurrence relationships:

$$\beta_{j+1} \mathbf{q}_{j+1} = \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j - \alpha_j \mathbf{q}_j - \beta_j \mathbf{q}_{j-1} = \mathbf{r}_j \quad (11)$$

$$\bar{\beta}_{j+1} \bar{\mathbf{q}}_{j+1} = \mathbf{K}^{-T} \mathbf{M}^T \bar{\mathbf{q}}_j - \bar{\alpha}_j \bar{\mathbf{q}}_j - \bar{\beta}_j \bar{\mathbf{q}}_{j-1} = \bar{\mathbf{r}}_j \quad (12)$$

and

$$\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_j] \quad (13)$$

$$\bar{\mathbf{Q}} = [\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2, \dots, \bar{\mathbf{q}}_j] \quad (14)$$

We also enforce the \mathbf{K} -orthogonality of the Lanczos vectors $\mathbf{q}_1, \dots, \mathbf{q}_j$ and $\bar{\mathbf{q}}_1, \dots, \bar{\mathbf{q}}_j$, i.e.

$$\bar{\mathbf{q}}_i^T \mathbf{K} \mathbf{q}_j = \delta_{ij} \quad \text{or} \quad \bar{\mathbf{Q}}^T \mathbf{K} \mathbf{Q} = \mathbf{I} \quad (15)$$

where δ_{ij} is the Kronecker delta and \mathbf{I} the unit matrix. Combining (11), (12) and (15) we obtain the following important relations:

$$\alpha_j = \bar{\alpha}_j = \bar{\mathbf{q}}_j^T \mathbf{M} \mathbf{q}_j \quad (16)$$

$$\beta_{j+1} = \bar{\beta}_{j+1} = (\bar{\mathbf{r}}_j^T \mathbf{K} \mathbf{r}_j)^{1/2} \quad (17)$$

and

$$\bar{\mathbf{q}}_i^T \mathbf{M} \mathbf{q}_j = \beta_{j+1} \delta_{i,j+1} + \alpha_j \delta_{i,j} + \beta_j \delta_{i,j-1} \quad (18)$$

When the recurrence relations (11) and (12) are used to generate \mathbf{Q} and $\bar{\mathbf{Q}}$ according to (13) and (14), the reduced eigenproblems (10) take the following simple forms:

$$\mathbf{T}\mathbf{s} = \mu \mathbf{s} \quad \text{and} \quad \bar{\mathbf{s}}^T \mathbf{T} = \mu \bar{\mathbf{s}}^T \quad (19)$$

where \mathbf{T} is a symmetric tridiagonal matrix

$$\mathbf{T} = \begin{bmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_1 & \beta_3 & & \\ & & \ddots & & \\ & & & \beta_{j+1} & \alpha_j \end{bmatrix} \quad (20)$$

The α s and β s are defined by (16) and (17). There is a variant of the Lanczos algorithm that produces an unsymmetric tridiagonal matrix, but it has been shown [Reference 18, p. 268] to be more sensitive to round-off errors than the present symmetric version. On the other hand, the present version introduces complex numbers into the problem. Considering (17), $\bar{\mathbf{r}}_j^T \mathbf{K} \mathbf{r}_j$ could be a negative quantity, resulting in a pure imaginary β_{j+1} , and \mathbf{T} is, in general, a complex matrix. Owing to the

symmetry of \mathbf{T} , the left and right eigenvectors of the reduced eigenproblems (19) are identical; this means

$$\mathbf{s} = \bar{\mathbf{s}} \quad (21)$$

Finally, the general Lanczos algorithm can be summarized as follows: choose $\mathbf{q}_0 = \bar{\mathbf{q}}_0 = \mathbf{0}$ and two arbitrary starting vectors \mathbf{r}_0 and $\bar{\mathbf{r}}_0$. Let $\beta_1 = (\bar{\mathbf{r}}_0^T \mathbf{K} \mathbf{r}_0)^{1/2}$. Use the recurrence relations to calculate the Lanczos vectors:

$$\begin{aligned} \mathbf{q}_j &= \mathbf{r}_{j-1}/\beta_j, & \bar{\mathbf{q}}_j &= \bar{\mathbf{r}}_{j-1}/\beta_j \\ \alpha_j &= \bar{\mathbf{q}}_j^T \mathbf{M} \mathbf{q}_j \\ \mathbf{r}_j &= \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j - \alpha_j \mathbf{q}_j - \beta_j \mathbf{q}_{j-1} \\ \bar{\mathbf{r}}_j &= \mathbf{K}^{-T} \mathbf{M}^T \bar{\mathbf{q}}_j - \alpha_j \bar{\mathbf{q}}_j - \beta_j \bar{\mathbf{q}}_{j-1} \\ \beta_{j+1} &= (\bar{\mathbf{r}}_j^T \mathbf{K} \mathbf{r}_j)^{1/2} \end{aligned}$$

After j steps, the eigenvalues of the complex tridiagonal matrix \mathbf{T} should be close approximations of the eigenvalues of the original system (7), and its eigenvectors are found by (9).

3. APPLICATION OF THE GENERAL ALGORITHM TO GYROSCOPIC SYSTEMS

The algorithm presented in Section 2 does not take advantage of any specific features of the matrices \mathbf{K} and \mathbf{M} . When the matrix \mathbf{K} is symmetric and positive definite, and \mathbf{M} skew-symmetric as is the case for gyroscopic systems, the algorithm simplifies considerably. In what follows, we will use the notation 'Re' for real quantities, and 'Im' for pure imaginary quantities. We start the procedure by choosing $\mathbf{r}_0 = \bar{\mathbf{r}}_0$ to be an arbitrary Im vector, so that $\beta_1 = (\bar{\mathbf{r}}_0^T \mathbf{K} \mathbf{r}_0)^{1/2}$ must be Im, since \mathbf{K} is positive definite. Proceeding now with the first step of the algorithm, we find

$$\begin{aligned} \mathbf{q}_1 &= \mathbf{r}_0/\beta_1, & \bar{\mathbf{q}}_1 &= \mathbf{q}_1; & \mathbf{q}_1 & \text{is a Re vector} \\ \alpha_1 &= \bar{\mathbf{q}}_1^T \mathbf{M} \mathbf{q}_1 = 0; & & & \text{since } \mathbf{M} & \text{is skew-symmetric} \\ \mathbf{r}_1 &= -\bar{\mathbf{r}}_1; & & & \text{since } \mathbf{M} &= -\mathbf{M}^T, \mathbf{r}_1 \text{ is a Re vector} \\ \beta_2 &= (\bar{\mathbf{r}}_1^T \mathbf{K} \mathbf{r}_1)^{1/2}; & & & \beta_2 & \text{is Im, since } \mathbf{K} \text{ is positive definite} \end{aligned}$$

In the second step, similar results are found:

$$\begin{aligned} \mathbf{q}_2 &= \mathbf{r}_1/\beta_2, & \bar{\mathbf{q}}_2 &= -\mathbf{q}_2; & \mathbf{q}_2 & \text{is an Im vector} \\ \alpha_2 &= \bar{\mathbf{q}}_2^T \mathbf{M} \mathbf{q}_2 = 0; & & & \text{since } \mathbf{M} & \text{is skew-symmetric} \\ \mathbf{r}_2 &= \bar{\mathbf{r}}_2; & & & \text{since } \mathbf{M} &= -\mathbf{M}^T, \mathbf{r}_2 \text{ is an Im vector} \\ \beta_3 &= (\bar{\mathbf{r}}_2^T \mathbf{K} \mathbf{r}_2)^{1/2}; & & & \beta_3 & \text{is Im since } \mathbf{K} \text{ is positive definite.} \end{aligned}$$

The pattern observed in the two first steps of the algorithm will keep on repeating, yielding the following results:

$$\text{all } \alpha_i = 0 \quad (22)$$

$$\mathbf{q}_1, \mathbf{q}_3, \mathbf{q}_5 \dots \text{ are Re, and } \mathbf{q}_2, \mathbf{q}_4, \mathbf{q}_6 \dots \text{ are Im} \quad (23)$$

$$\mathbf{Q} = [\mathbf{q}_1, -\mathbf{q}_2, \mathbf{q}_3, -\mathbf{q}_4 \dots] \quad (24)$$

$$\text{all } \beta_i \text{ are Im} \quad (25)$$

Relations (20), (22) and (25) show that \mathbf{T} is an Im matrix; hence the reduced eigenproblem (19)

becomes

$$\mathbf{T}'\mathbf{s} = \mu'\mathbf{s} \quad (26)$$

where $\mathbf{T}' = \mathbf{T}/i$ and $\mu' = \mu/i$. \mathbf{T}' is now a Re, symmetric, tridiagonal matrix and its eigenvalues μ' are of course Re, occurring in pairs of opposite signs, as proved in the Appendix. This means that the eigenvalues μ are Im and appear in conjugate pairs (this is a well known property of gyroscopic systems¹³). Relations (9) and (32) show that the eigenvectors corresponding to conjugate μ s are complex conjugates, as expected since \mathbf{K} and \mathbf{M} are Re. Finally relations (9), (21), (13) and (24) combine to yield

$$\bar{\mathbf{y}} = \mathbf{y}^* \quad (27)$$

where $(\)^*$ denotes a complex conjugate quantity. This means that the left eigenvectors are the complex conjugates of the right eigenvectors. (This is again a well known property of gyroscopic systems.¹³)

The two most significant simplifications of the general algorithm are (1). The system of left Lanczos vectors $\bar{\mathbf{Q}}$ is identical to the system of right vectors \mathbf{Q} , except for a sign change [see (13) and (24)]. Hence, only the right system only needs to be calculated, halving the arithmetic and storage requirements. (2) Even though Re and Im quantities appear in the algorithm, no complex arithmetic is required.

It is also important to note that full advantage is taken of the specific nature of gyroscopic systems, and all their properties are found as natural consequences of the calculation process. The simplified algorithm is summarized as follows.

Choose $\mathbf{q}_0 = \mathbf{0}$ and an arbitrary Im starting vector \mathbf{r}_0 . Set $\beta_1 = (\mathbf{r}_0^T \mathbf{K} \mathbf{r}_0)^{1/2}$. Use the recurrence relations to generate the Lanczos vectors:

$$\begin{aligned} \mathbf{q}_j &= \mathbf{r}_{j-1} / \beta_j \\ \mathbf{r}_j &= \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j - \beta_j \mathbf{q}_{j-1} \\ \beta_{j+1} &= (\mathbf{r}_j^T \mathbf{K} \mathbf{r}_j)^{1/2} \end{aligned}$$

4. SELECTIVE ORTHOGONALIZATION

The algorithm described in the two previous sections is not suitable for practical implementations. Indeed, it was recognized very early^{6,17-19} that with this simple algorithm, the orthogonality of the Lanczos vectors (15) rapidly deteriorates due to round-off errors. The reasons for this loss of orthogonality are the repeated multiplications by $\mathbf{K}^{-1} \mathbf{M}$ in (11) to find the successive Lanczos vectors. In fact these repeated multiplications are the basis of the power method,¹⁷ which is known to produce the dominant eigenvector, starting with an arbitrary vector. In the Lanczos process, the repeated multiplications amplify the components of round-off error which are not orthogonal to the dominant eigenvector, resulting in a rapid loss of orthogonality, and hence the algorithm keeps on recalculating copies of previously found eigenvectors.¹⁸

Lanczos proposed an explicit reorthogonalization of \mathbf{q}_j against all previous \mathbf{q}_s . Although this improves matters significantly,^{6,19} there is a strong incentive to avoid the storage and arithmetic cost of these reorthogonalizations.

Selective orthogonalization is now a well-developed technique that maintains a strong linear independence among the Lanczos vectors. A detailed analysis of the procedure can be found in the literature.^{5,18} In the case of gyroscopic systems the eigenvectors are complex, so that selective orthogonalization must be performed against both the Re and Im parts of the converged eigenvectors, but here again complex arithmetic can be avoided.

5. NUMERICAL EXAMPLES

The numerical procedure described in the previous sections has been applied to two specific examples. First, the well-known problem of the free vibration analysis of a rotating beam is considered. Hodges and Rutkowski⁹ have given an exact solution of this problem that will be used here as a reference.

A uniform beam is rotating at a constant angular speed Ω about an axis fixed in space. Vibration modes corresponding to pure out-of-plane motion of the beam are called 'flapping modes', whereas pure in-plane vibrations are 'lead-lag modes'. The mass per unit span of the beam $m = 0.113$ kg/m, the flapping bending stiffness $EI_f = 2.45$ Nm², the lead-lag bending stiffness $EI_l = 36.3$ Nm² and the length $L = 0.508$ m.

The matrices of the eigenproblem (4) were generated using three four-noded beam finite elements.²⁰ Table I lists the dimensionless natural frequencies μ , versus the dimensionless angular speed λ defined as

$$\mu^2 = \omega^2 \frac{mL^4}{EI}; \quad \lambda^2 = \Omega^2 \frac{mL^4}{EI} \quad (28)$$

For flapping modes the flapping bending stiffness is used in (28), whereas the lead-lag bending stiffness is used for the lead-lag modes.

The flapping frequencies are found to be in excellent agreement with the reference solution. The agreement of the lead-lag frequencies is good, but for very high angular speed slightly lower frequencies are predicted by the present solution, as it takes into account the elongation of the beam under centrifugal loading, whereas the reference solution assumes an inextensible beam.

The second example is the analysis of an L-shaped uniform cantilevered beam. Figures 1 and 2 describe the geometry of the beam as well as the relevant parameters. The three lowest frequencies of the system are listed in Table II for various values of the angular speed Ω . The stiffening of the beam under spin induced loads, the large deflections and rotations of the beam in the steady state equilibrium position and the gyroscopic effects all combine to influence the natural frequencies of the structure for increasing angular speed.

This problem was also used to assess the computational efficiency of the Lanczos algorithm. For zero angular speed, the eigenproblem has the form (2) and was solved using the subspace iteration method to yield the 'non-rotating modes'. For non-zero angular speed ('rotating modes'), the eigenproblem has the form (4) and was solved using the present Lanczos algorithm. Note that the matrices involved in both eigenproblems have the same size and bandwidth. The CPU times (in seconds) required for the calculation of 6 eigenvalues and the corresponding eigenvectors are given in Table III.

Table I. Dimensionless frequency μ versus dimensionless angular speed λ

λ	1st flapping mode		1st lead-lag mode		2nd flapping mode	
	Present	Reference	Present	Reference	Present	Reference
0	3.516	3.516	3.516	3.516	22.03	22.03
2	4.138	4.137	3.617	3.622	22.61	22.61
4	5.587	5.585	3.877	3.898	24.28	24.27
6	7.364	7.360	4.229	4.263	26.83	26.81
8	9.261	9.257	4.607	4.657	30.03	30.00
10	11.21	11.20	4.976	5.049	33.70	33.64
12	13.17	13.17	5.326	5.427	37.73	37.60

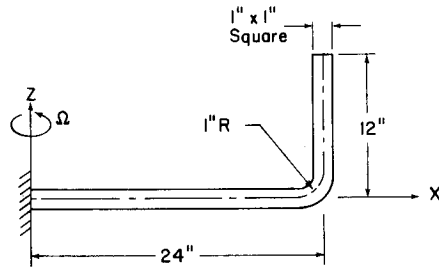


Figure 1. Sketch of a cantilever L-shaped rotating beam: bending stiffness = $0.83 \times 10^6 \text{ lb-in}^2$; torsional stiffness = $0.67 \times 10^6 \text{ lb-in}^2$; mass per unit length = $1.575 \text{ lb-s}^2/\text{in}^2$

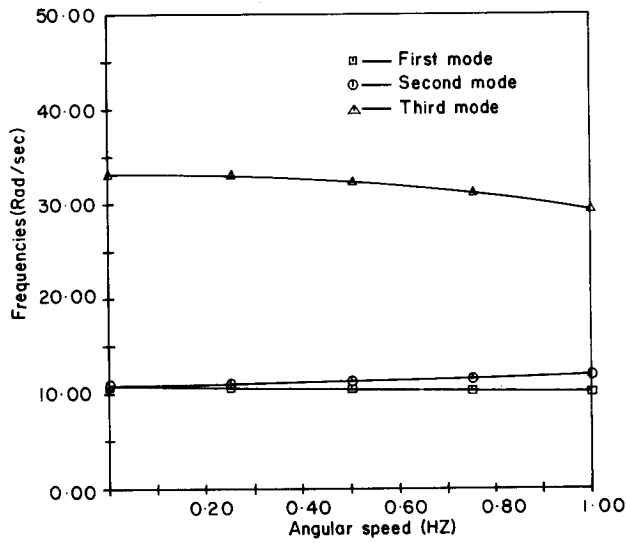


Figure 2. Vibration frequencies against angular speed Ω

Table II. Natural frequency $\omega(\text{rad/s})$ versus angular speed $\Omega(\text{Hz})$

Ω	1st mode	2nd mode	3rd mode
0.0	10.735	10.872	33.168
0.25	10.582	11.052	32.957
0.50	10.416	11.310	32.304
0.75	10.297	11.600	31.144
1.00	10.196	11.968	29.323

Table III

	Size of the problem (n)	
	72	126
'Non-rotating modes' using subspace iteration	1.95	4.53
'Rotating modes' using Lanczos algorithm	1.02	1.47

The Lanczos algorithm appears to be significantly more efficient than the subspace iteration method, even for the solution of a more complex problem. In other words, the calculation of 'rotating modes' with the Lanczos algorithm is cheaper than the calculation of 'non-rotating modes' with the subspace iteration method. Of course, 'non-rotating modes' could also be calculated with the standard Lanczos algorithm, which has been shown²¹ to be more efficient than the subspace iteration method. Since no complex arithmetic is required in the present Lanczos algorithm, the number of major multiplications involved in the procedure is the same as that reported in Reference 21.

6. CONCLUSIONS

A variant of the general Lanczos algorithm has been presented which yields an efficient solution of the eigenproblem arising in the analysis of spinning structures. The algorithm takes full advantage of the specific nature of gyroscopic problems, resulting in two important simplifications: (1) only the right Lanczos vectors need to be generated, and (2) complex arithmetic is completely avoided. The Lanczos method affords an accurate, cost-effective calculation of the natural frequencies of spinning structures ('rotating frequencies').

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APPENDIX

In Section 3, the eigenvalues μ' of the reduced system (26) were shown to be the roots of the following determinant, at step j :

$$D_j = \det \begin{bmatrix} -\mu' & \beta_2 \\ \beta_2 - \mu' \beta_3 & \\ & \vdots \\ & \beta_j - \mu' \end{bmatrix} \quad (29)$$

Elementary determinant theory yields the following recurrence relation:

$$D_j = -\mu' D_{j-1} - \beta_j^2 D_{j-2} \quad (30)$$

It is now easy to see that D_j is an even function of μ' for even j , whereas for odd j , D_j is an even

function of μ' , multiplied by μ' . This comes from the recurrence (30) and the fact that this property holds for D_2 and D_3 . It follows that the eigenvalues μ' occur in pairs of opposite signs, since they are the roots of an even function. The eigenvalue $\mu' = 0$ appears at each odd step, but has no physical meaning.

Finally, simple calculations show that if

$$(S_1, S_2, S_3, S_4 \dots) \quad (31)$$

is the eigenvector corresponding to μ' , the eigenvector corresponding to the eigenvalue $-\mu'$ is

$$(S_1, -S_2, S_3, -S_4 \dots) \quad (32)$$

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