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Space-Time Perturbation Modes for Non-Linear Dynamic Analysis of Beams

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Abstract. In this work an attempt is made at bridging the powerful perturbation methods of analytical dynamics to the versatile finite element techniques which can readily handle arbitrarily complex structures. The proposed analysis methodology has two distinguishing features. First, a space-time finite element formulation is used, and hence the concept of modes is here naturally extended to that of space-time modes, where the time dependency is implied in the assumed modes. As a result, the partial differential equations of motion are directly reduced to purely algebraic non-linear simultaneous equations. Second, perturbation modes, rather than the usual vibration mode shapes are used and shown to be an appropriate basis for non-linear dynamic analysis. These modes bring information about the non-linearities of the system through the higher order derivatives of the strain and kinetic energies. The procedure is illustrated on non-linear beam problems and the results are compared with those of a full finite element model, i.e., when all the degrees of freedom are considered, as well as with analytical results, when available.

Key words: Beams, non-linear dynamics, perturbation modes.

Introduction

There are two quite distinct approaches to non-linear dynamic analysis. The first is the classical analytical approach in which the non-linear partial differential equations of motion of the structure are derived, then various analytical techniques are used to predict the response as well as stability boundaries for the motion. On the other hand, purely numerical techniques, such as the finite element method, have gained popularity in the last few decades. It is important to realize that these approaches are not competing with each other, but are in fact complementary.

Indeed, the finite element method can treat arbitrarily complex structures involving both material and geometric non-linearities; however, the solution phase generally consists only of the time integration of the resulting equations of motion. In other words the numerical solution merely simulates, as accurately as possible, the physical system. It gives the response of the system under a given loading but gives no information about why this response is what it is. On the other hand, analytical techniques can only handle very simplified structural configurations, and more often than not, give an approximate prediction of the response. However, the analytical nature of the solution technique readily allows the determination of key parameters that will characterize the nature of the response, define zones where the non-linear effects will be prominent and identify stable and unstable regions. In other

words, analytical techniques give invaluable insight into the non-linear dynamic behavior of structures.

This paper is an attempt at bridging the two disciplines. The goal is to retain the versatility of finite element techniques, while using the perturbation methods characteristic of analytical dynamics. The methodology developed here is applied to the non-linear dynamic modal analysis of beams. To this purpose, a mixed non-linear beam model is discussed in the next sections. With the help of Hamilton's Principle, a space-time finite element discretization is readily obtained, which allows the treatment of both initial value and periodic problems within the same general framework.

Particular attention must be devoted in selecting a methodology for the measure of finite rotations, if a modal reduction approach is to be developed. Indeed, the evaluation of the coefficients of the modal expansion requires the computation of integrals containing finite rotation expressions. If the finite rotation representation involves purely polynomial expressions only, the integrals are readily computed. However, if the finite rotation representation involves transcendental functions, these integrals can only be evaluated as a truncated series, resulting in an approximate treatment of the finite rotations. While wide choice of different parameterizations is available, polynomial expressions are a unique feature of quaternions. However, the adoption of quaternions is not without drawbacks, since the unitary quaternion constraint must be introduced in the formulation, thus increasing the computational cost. For these reasons, two different parameterizations of finite rotations have been investigated in this work, namely the conformal rotation vector and the Euler parameters or quaternions. Both are discussed in the following sections, together with a description of a suitable Lagrange multiplier method for enforcing the unitary quaternion condition.

In the context of space-time finite element formulations, the concept of static modes is here extended to that of space-time modes. When these latter modes are used, the time dependency of the solution is implied in the assumed mode shapes and thus the reduced equations in the generalized coordinates are simply algebraic equations.

The choice of an appropriate basis for non-linear dynamic modal analysis is crucial for achieving accuracy and reliability. In particular, it is well known that the natural modes behave poorly in the context of highly non-linear problems. In this work the perturbation modes have been selected as modal basis. It is shown how they extract information about the non-linear behavior of the structure from the higher order derivatives of the strain and kinetic energies.

The effectiveness of the formulation presented here is illustrated by means of simple but meaningful example problems related to the periodic response of beams with different boundary conditions subjected to sinusoidal loads. While the results obtained are in excellent agreement with analytical solutions found in the literature, it is shown how some phenomena such as the shear effects, are naturally handled in the context of finite element based modal reduction approaches.

Non-Linear Beam Space-Time Finite Element Formulation

The non-linear beam model used in this work is similar to that found in [1, 2], and it is summarized here for convenience in order to introduce the appropriate notation. In this model, displacements, momenta and forces all represent independent fields. The rational for adopting a mixed formulation lies in the fact that in this way the stress-strain and momentum-velocity relations are only weakly enforced. Thus errors in the strains or the velocities do not necessarily result in large errors in forces and momenta. Moreover, the order of the non-

linearities appearing in mixed models is much lower than in displacement based approaches. The resulting formulation is well suited for modal analysis [6].

Let $\mathcal{S}_0$ be an inertial frame and $\mathcal{S}$ a frame of unit vectors $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ attached to the beam in the undeformed straight and unloaded configuration. The pair $(\mathbf{s}_2, \mathbf{s}_3)$ defines the plane of the cross-section and $\mathbf{s}_1$ the tangent to the reference line. The local triad $\mathcal{S}^*$ identifies the corresponding quantities in the deformed configuration. The orientation of the triad $\mathcal{S}$ with respect to $\mathcal{S}_0$ is determined by a known rotation tensor which components in $\mathcal{S}$ are denoted $\mathbf{R}_0$. The orientation of $\mathbf{S}^*$ with respect to $\mathcal{S}$ is determined by an unknown rotation tensor which components in $\mathcal{S}$ are denoted $\mathbf{R}$. The displacement of the reference line is determined by an unknown vector which components in $\mathcal{S}$ are $\mathbf{u}$.

Given these definition, the Lagrangian of a beam element whose cross-section does not deform in its own plane writes

$$L = T - U = \int_0^L \left(\frac{1}{2} \mathbf{v}^{*T} \mathbf{M}^* \mathbf{v}^* - \frac{1}{2}\, \mathbf{e}^{*T} \mathbf{C}^* \mathbf{e}^* \right) \mathrm{d}l. \tag{1}$$

The matrices $\mathbf{M}^*$ and $\mathbf{C}^*$ identify the mass and stiffness properties of the beam cross-section in $\mathcal{S}^*$. The components of the angular and linear velocity vectors of the cross-section in $\mathcal{S}^*$ are noted $\mathbf{v}^*$ and can be expressed as

$$\mathbf{v}^* = \begin{bmatrix} \mathbf{R}_0^T \mathbf{R}^T & 0 \\ 0 & \mathbf{R}_0^T \mathbf{R}^T \end{bmatrix} \begin{Bmatrix} \omega \\ \dot{\mathbf{u}} \end{Bmatrix} = \mathcal{R}^T \begin{Bmatrix} \omega \\ \dot{\mathbf{u}} \end{Bmatrix}.$$

ω are the components of the angular velocity vector in $\mathcal{S}$. The components of the sectional curvature and strain vectors in $\mathcal{S}^*$ are noted $\mathbf{e}^*$ and can be expressed as

$$\mathbf{e}^* = \begin{bmatrix} \mathbf{R}_0^T \mathbf{R}^T & 0 \\ 0 & \mathbf{R}_0^T \mathbf{R}^T \end{bmatrix} \begin{Bmatrix} \mathbf{k} \\ \overline{\mathbf{u}}' \end{Bmatrix} - \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \mathcal{R}^T \begin{Bmatrix} \mathbf{k} \\ \overline{\mathbf{u}}' \end{Bmatrix} - \begin{Bmatrix} 0 \\ 1 \end{Bmatrix},$$

where $\mathbf{k}$ are components of the curvature vector in $\mathcal{S}$. $\overline{\mathbf{u}}' = \mathbf{u}' + \mathbf{r}_{01}$, $\mathbf{r}_{01}$ being the first column of the rotation matrix $\mathbf{R}_0$, and $1^T = (1,0,0)$. The symbol $(\dot{}) = \partial(\ )/\partial t$ denotes partial time derivatives and $(\ )' = \partial(\ )/\partial l$ partial space derivatives.

A Hamiltonian function can now be defined as

$$H = \int_0^L (\mathbf{p}^{*T} \mathbf{v}^* - \mathbf{f}^{*T} \mathbf{e}^*)\, \mathrm{d}l - L, \tag{2}$$

where $\mathbf{p}^*$ and $\mathbf{f}^*$ are momenta and forces in the moving frame. The governing statement of the problem of motion is provided by the following three-field energy principle

$$\int_{t_1}^{t_2} \int_0^L \delta \left(\mathbf{p}^{*T} \mathbf{v}^* - \mathbf{f}^{*T} \mathbf{e}^* - \frac{1}{2}\, \mathbf{p}^{*T} \mathbf{M}^{*-1} \mathbf{p}^* + \frac{1}{2}\, \mathbf{f}^{*T} \mathbf{C}^{*-1} \mathbf{f}^* + \delta \mathbf{d}^T \mathbf{f}_e \right) \mathrm{d}l\, \mathrm{d}t$$

$$= \left(\int_0^L \mathbf{p}^{b^T} \delta \mathbf{d}\, \mathrm{d}l \right) \Bigg|_{t_1}^{t_2} - \left(\int_{t_1}^{t_2} \mathbf{f}^{b^T} \delta \mathbf{d}\, \mathrm{d}t \right) \Bigg|_0^L, \tag{3}$$

where $\mathbf{d} = (\mathbf{r}, \mathbf{u})$, and $\mathbf{r}$ is the set of parameters employed for the description of the finite rotations. $\mathbf{f}_e = (\mathbf{m}_e, \mathbf{s}_e)$ denotes the generalized external force vector, being $\mathbf{m}_e$ applied moments and $\mathbf{s}_e$ applied forces, while the symbol $(\,)^b$ denotes boundary quantities. In equation (3), displacements, momenta and forces represent three independent fields and hence the constitutive equations

$$\mathbf{p}^* = \mathbf{M}^*\mathbf{v}^*, \quad \mathbf{f}^* = \mathbf{C}^*\mathbf{e}^*$$

are only weakly enforced.

The expression of the angular velocity and curvature in the moving frame are respectively

$$\omega^* = \mathbf{R}^T\omega = \boldsymbol{\Gamma}^T(\mathbf{r})\dot{\mathbf{r}}$$

and

$$\mathbf{k}^* = \mathbf{R}^T\mathbf{k} = \boldsymbol{\Gamma}^T(\mathbf{r})\mathbf{r}'.$$

The expressions for the rotation tensor $\mathbf{R}$ and of the operator $\boldsymbol{\Gamma}$ depend on the particular parameters $\mathbf{r}$ adopted to measure the finite rotations.

The energy principle stated by equation (3) provides the basis for a space-time finite element approximation. When a small time interval is considered, the displacements and momenta at the end of the interval can be computed based on the knowledge of the corresponding quantities at the beginning of the interval, resulting in a time stepping process. On the other hand, if the entire time period of interest is considered, an assembly process can be developed to obtain the solution over the whole interval. The latter approach is taken in this effort, and the resulting non-linear algebraic equations can be linearized with the help of the Newton–Rapson method to yield

$$\begin{bmatrix} \mathbf{T}_{dd_{(e)}} & \mathbf{T}_{dp_{(e)}} & \mathbf{T}_{df_{(e)}} \\ \mathbf{T}^T_{dp_{(e)}} & \mathbf{T}_{pp_{(e)}} & 0 \\ \mathbf{T}^T_{df_{(e)}} & 0 & \mathbf{T}_{ff_{(e)}} \end{bmatrix} \begin{Bmatrix} \Delta\bar{\mathbf{d}}_{(e)} \\ \Delta\bar{\mathbf{p}}_{(e)} \\ \Delta\bar{\mathbf{f}}_{(e)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}_{(e)} \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \mathbf{R}_{d_{(e)}} \\ \mathbf{R}_{p_{(e)}} \\ \mathbf{R}_{f_{(e)}} \end{Bmatrix}, \tag{4}$$

where $\Delta\bar{\mathbf{d}}$, $\Delta\bar{\mathbf{p}}$ and $\Delta\bar{\mathbf{f}}$ are increments to displacement, momentum and force nodal values respectively, while the subscript $(\,)_{(e)}$ refers to elemental quantities and the symbol $(\bar{})$ denotes nodal quantities.

The continuity requirements implied by (3) allow to interpolate momenta and forces as discontinuous quantities across elements. They can thus be eliminated at the element level and recovered after convergence. The reduced elemental matrix problem thus writes

$$\bar{\bar{\mathbf{T}}}_{dd_{(e)}}\Delta\bar{\mathbf{d}}_{(e)} = \mathbf{Q}_{(e)} - \bar{\bar{\mathbf{R}}}_{d_{(e)}}, \tag{5}$$

where

$$\begin{aligned} \bar{\bar{\mathbf{T}}}_{dd_{(e)}} &= \mathbf{T}_{dd_{(e)}} - \mathbf{T}_{dp_{(e)}}\mathbf{T}^{-1}_{pp_{(e)}}\mathbf{T}^T_{dp_{(e)}} - \mathbf{T}_{df_{(e)}}\mathbf{T}^{-1}_{ff_{(e)}}\mathbf{T}_{df_{(e)}}, \\ \bar{\bar{\mathbf{R}}}_{d_{(e)}} &= \mathbf{R}_{d_{(e)}} + \mathbf{T}^{-1}_{pp_{(e)}}\mathbf{R}_{p_{(e)}} + \mathbf{T}^{-1}_{ff_{(e)}}\mathbf{R}_{f_{(e)}}. \end{aligned}$$

The global matrix problem can then be obtained through the standard finite element assembly process performed on the corresponding elemental matrices to yield

$$\mathbf{T}\Delta\mathbf{d} = \mathbf{Q} - \mathbf{R}. \tag{6}$$

The global matrix problem (6), can be used for solving both initial value and periodic problems. For the latter case, assuming a period of time $T = t_f - t_i$ and assembling a suitable number of time elements over the period T, the solution is obtained simply by enforcing the appropriate periodicity constraints on the displacements $\overline{\mathbf{d}}_i = \overline{\mathbf{d}}_f$ and on the momenta $\mathbf{p}_i^b = \mathbf{p}_f^b$. This can be readily achieved by folding the rows and columns relative to quantities at time t_f of equation (6) on the corresponding rows and columns relative to quantities at time t_i. The ability to solve both initial value and periodic problems within the same general framework, justifies the interest that the finite element in time method has raised.

Parameterization of Finite Rotations

A wide choice of different parameterizations of finite rotations is available to the analyst: different sets of Euler angles depending on the axis sequence, the finite rotation vector, the conformal rotation vector, Rodriguez parameters and quaternions or Euler parameters (cf. for example [7] and papers referenced therein). While the choice of a particular parameterization can be a matter of personal taste, specific parameterizations present distinct characteristics that might be advantageous for specific problems.

In this work two different approaches have been used to measure finite rotations: the conformal rotation vector (CRV) and the Euler parameters. These two approaches can be defined in terms of the finite rotation vector $\boldsymbol{\psi} = \theta \cdot \boldsymbol{\kappa}$, being $\boldsymbol{\kappa}$ the unit vector about which a rotation of magnitude θ occurs.

The quaternion $\mathbf{q} = (q_s, \mathbf{q}_v)$ whose scalar part is q_s and vector part is $\mathbf{q}_v$, is defined in terms of $\boldsymbol{\psi}$ as

$$q_s = \cos\frac{\theta}{2}, \quad \mathbf{q}_v = \sin\frac{\theta}{2}\,\boldsymbol{\kappa}.$$

Clearly, the four elements of the quaternion $\mathbf{q}$ are not independent as they are related by the constraint relation

$$\phi = \mathbf{q}^T\mathbf{q} - 1 = 0. \tag{7}$$

Quaternions present the unique characteristic of being free from any singularity of representation, whereas other representations involve particular rescaling or updating techniques to avoid such singularities. However, their use is in general computationally expensive since four parameters are used instead of three. Moreover, if the unitary quaternion constraint is enforced by means of the Lagrange multiplier method, one additional degree of freedom is introduced for each quaternion.

The rotation tensor in terms of quaternions can be expressed as

$$\mathbf{R} = \mathbf{I} + 2q_s\mathbf{q}_v \times \mathbf{I} + 2\mathbf{q}_v \times \mathbf{q}_v \times \mathbf{I} = \mathbf{B}_\perp^T(\mathbf{q})\mathbf{A}_\perp(\mathbf{q})$$

and the operator $\boldsymbol{\Gamma}$ is given by

$$\boldsymbol{\Gamma} = \mathbf{A}_\perp(\mathbf{q}).$$

The operators

$$\mathbf{A}_\perp(\mathbf{q}) = \begin{bmatrix} -\mathbf{q}_v^T \\ q_s\mathbf{I} + \mathbf{q}_v \times \mathbf{I} \end{bmatrix},$$

$$\mathbf{B}_\perp(\mathbf{q}) = \begin{bmatrix} -\mathbf{q}_v^T \\ q_s\mathbf{I} - \mathbf{q}_v \times \mathbf{I} \end{bmatrix}$$

are normal to each quaternion $\mathbf{q}$, since

$$\mathbf{q}^T\mathbf{A}_\perp(\mathbf{q}) = 0, \quad \mathbf{q}_T\mathbf{B}_\perp(\mathbf{q}) = 0 \quad \forall \mathbf{q}.$$

The conformal rotation vector is related to Euler parameters by the following conformal mapping

$$\begin{aligned}(a_s, \mathbf{a}_v) &= \frac{4}{1+q_s}(q_s, \mathbf{q}_v),\\ (q_s, \mathbf{q}_v) &= \frac{1}{4-a_s}(a_s, \mathbf{a}_v).\end{aligned}$$

Hence $\mathbf{a}_v = 4\tan(\theta/4)\boldsymbol{\kappa}$, and $a_s = 2 - \mathbf{a}_v^T\mathbf{a}_v/8$. The rotation tensor $\mathbf{R}$ and the operator $\boldsymbol{\Gamma}$ become

$$\begin{aligned}\mathbf{R} &= \frac{1}{(4-a_s)^2}\left((a_s^2 - \mathbf{a}_v^T\mathbf{a}_v)\mathbf{I} + 2a_s\mathbf{a}_v \times \mathbf{I} + 2\mathbf{A}\,\mathrm{diag}\,(\mathbf{a}_v)\right),\\ \boldsymbol{\Gamma} &= \frac{2}{(4-a_s)^2}\left(a_s\mathbf{I} + \mathbf{a}_v \times \mathbf{I} + \frac{1}{4}\mathbf{A}\,\mathrm{diag}\,(\mathbf{a}_v)\right),\end{aligned}$$

where

$$\mathbf{A} = [\mathbf{a}_v \mid \mathbf{a}_v \mid \mathbf{a}_v]$$

and

$$\mathrm{diag}\,(\mathbf{a}_v) = \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{bmatrix}.$$

Note that a_s is introduced only with the purpose of simplifying the notation and thus it is not a redundant parameter. The conformal rotation vector presents no singularities within the range $-\pi \leq \theta \leq \pi$. However, in order to avoid limitations on θ, a rescaling approach can be effectively adopted [3].

It is clear that the expressions of the operators $\mathbf{R}$ and $\boldsymbol{\Gamma}$ in terms of the conformal rotation vector involve rational fractions. On the contrary, if the Euler parameters are adopted, all the rotational operators are purely algebraic and polynomial. This unique feature allows performing a modal expansion without resorting to the moderate rotation assumption, as will be shown in the following section. Both representations will be used in this work since they seem to be in some way complementary: the conformal rotation vector is simpler and cheaper from a computational point of view but can only treat moderate rotations in the context of a modal reduction method, while the quaternions are much more expensive but can handle arbitrarily large rotations.

For the quaternion $\mathbf{q}$ to represent a rotation, the unique quaternion condition (7) must be introduced in the energy principle (3). Two basic approaches can be followed in order to enforce the constraint: the penalty method and the Lagrange multiplier method. The penalty method is based on assigning a high cost to the violation of the constraint. Clearly, as the cost goes to infinity the violation tends to zero. However, the choice of the value to assign to the penalty terms can be quite troublesome. In fact two contrasting requirements must be met: if on the one hand the penalty term should be chosen as large as possible, on the other hand the introduction of too large a number can cause ill conditioning of the tangent matrix,

loss of accuracy in the solution process as well as convergence difficulties. The problem is exacerbated when the penalty approach is adopted in the context of a reduction method. The delicate underlying structure of the physical system which is condensed in a small-size set of equations during the reduction process, is often irremediably destroyed by the large penalty numbers.

For these reasons, a Lagrangian multiplier method has been preferred in this work to the penalty approach. In the augmented Lagrangian method [4], the Lagrangian $L(\mathbf{d})$ in displacement based form for a system subject to the constraint (7) $\phi = 0$ is modified to write

$$\tilde{L}(\mathbf{d}) = L(\mathbf{d}) + \lambda\phi + \frac{1}{2}\, p\phi^2. \tag{8}$$

In equation (8), the scalar p is a penalty-like coefficient. However, it does not need to be a large number, since its only role is that of preventing the appearance of null pivots during the factorization process, thus allowing the use of linear system solvers without pivoting strategies.

The augmented Lagrangian method stated by (8), can be effectively extended to space-time formulations for dealing with the unitary quaternion constraints. In analogy with the scalar part of the quaternion $\mathbf{q}_s$, a "scalar" angular velocity can be defined as $\omega_s^* = 2\mathbf{q}^T\dot{\mathbf{q}}$ and analogously a "scalar" curvature as $k_s^* = 2\mathbf{q}^T\mathbf{q}'$. The modified Lagrangian for the constrained problem in mixed four-field form thus writes

$$\begin{aligned}\tilde{L} = \int_0^L \Big(&\mathbf{p}^{*T}\mathbf{v}^* - \mathbf{f}^{*T}\mathbf{e}^* - \frac{1}{2}\,\mathbf{p}^{*T}\mathbf{M}^{*-1}\mathbf{p}^* + \frac{1}{2}\,\mathbf{f}^{*T}\mathbf{C}^{*-1}\mathbf{f}^* \\ &+ \lambda(\mathbf{q}^T\mathbf{q} - q) + \frac{1}{2}\, m_s w_s^{*2} - \frac{1}{2}\, c_s k_s^{*2}\Big)\, \mathrm{d}l,\end{aligned} \tag{9}$$

where the independent fields are now displacements, momenta, forces and multipliers, and m_s and c_s are two scalar constants which correspond to the coefficient p of equation (8). It is convenient to define a four-dimensional angular velocity vector $\omega_4^* = (\omega_s^*, \boldsymbol{\omega}^*)$ and curvature vector $\mathbf{k}_4^* = (k_s^*, \mathbf{k}^*)$. Analogously, the part of the momentum and force vectors associated with the rotational degrees of freedom are expanded as four-dimensional vectors. Defining the expanded mass matrix

$$\mathbf{M}_\tau^* = \begin{bmatrix} m_s & 0 \\ 0 & \mathbf{M}^* \end{bmatrix}$$

and stiffness matrix

$$\mathbf{C}_\tau^* = \begin{bmatrix} c_s & 0 \\ 0 & \mathbf{C}^* \end{bmatrix},$$

the constrained energy principles writes more concisely

$$\begin{aligned}&\int_{t_1}^{t_2}\int_0^L \delta\left(\mathbf{p}_7^{*T}\mathbf{v}_7^* - \mathbf{f}_7^{*T}\mathbf{e}_7^* - \frac{1}{2}\,\mathbf{p}_7^{*T}\mathbf{M}_7^{*-1}\mathbf{p}_7^* + \frac{1}{2}\,\mathbf{f}_7^{*T}\mathbf{C}_7^{*-1}\mathbf{f}_7^* + \lambda(\mathbf{q}^T\mathbf{q} - 1)\right)\, \mathrm{d}l\, \mathrm{d}t \\ &\quad = \left(\int_0^L \mathbf{p}^{bT}\delta\mathbf{d}\, \mathrm{d}l\right)\Bigg|_{t_1}^{t_2} - \left(\int_{t_1}^{t_2} \mathbf{f}^{bT}\delta\mathbf{d}\, \mathrm{d}t\right)\Bigg|_0^L,\end{aligned} \tag{10}$$

where the subscripts refer to the actual dimensions of the vectors and matrices.

The continuity requirements implied by equation (10) allow one to interpolate the multipliers λ as discontinuous quantities across elements, exactly as momenta and forces. However, under certain boundary conditions this interpolation is inappropriate since it generates a singular integration scheme [5]. A simple remedy to this pathological behavior is to interpolate the multipliers as continuous quantities, in the same way as the displacements and rotation parameters.

Space-Time Modes and Modal Expansions

A modal expansion can be obtained from the energy principle (3). The independent displacement, force and momentum fields can be appropriately described by the following expansions about a given reference configuration

$$\mathbf{d} = \mathbf{d}^0 + \mathbf{d}^j \cdot \psi_d^j, \tag{11}$$

$$\mathbf{f}^* = \mathbf{f}^{*0} + \mathbf{f}^{*j} \cdot \psi_f^j, \tag{12}$$

$$\mathbf{p}^* = \mathbf{p}^{*0} + \mathbf{p}^{*j} \cdot \psi_p^j, \tag{13}$$

where $\mathbf{d}^0$, $\mathbf{f}^{*0}$ and $\mathbf{p}^{*0}$ are displacements, forces and momenta, relative to the reference equilibrium configuration. The assumed modes are $\mathbf{d}^j$, $\mathbf{f}^{*j}$ and $\mathbf{p}^{*j}$ for displacements, forces and momenta respectively, while the modal amplitudes or generalized coordinates are ψ_d^i, ψ_f^i and ψ_p^i.

In a classic modal approach [6], the modes $\mathbf{d}^j$, $\mathbf{f}^{*j}$ and $\mathbf{p}^{*j}$ are static modes, i.e., the generalized coordinates $\psi_d^i = \psi(t)_d^i$, $\psi_f^i = \psi(t)_f^i$ and $\psi_p^i = \psi(t)_p^i$ are functions of time.

However, in the context of space-time finite element formulations it seems natural to extend the concept of static modes to that of space-time modes. In this way the generalized coordinates are no longer functions of time, since the time dependency is included in the modal basis. This approach has several advantages. The classical procedure which makes use of static modes leads to a set of non-linear ordinary differential equations which must then integrated with respect to time, whereas using space-time modes a set of non-linear algebraic equations in the generalized coordinates is obtained. This set of equations directly provides the solution to the problem at a very low computational cost.

The reduced modal approximation is obtained by introducing the modal representation (12) into the Lagrangian to yield

$$L = T - U,$$

where

$$\begin{aligned} T = {} & T^0 + T_d^j \psi_d^j + T_p^j \psi_p^j + T_{dd}^{jk} \psi_d^j \psi_d^k + T_{dp}^{jk} \psi_d^j \psi_p^k + T_{pp}^{jk} \psi_p^j \psi_p^k \\ & + T_{ddd}^{jkl} \psi_d^j \psi_d^k \psi_d^l + T_{ddp}^{jkl} \psi_d^j \psi_d^k \psi_p^l + T_{dddd}^{jklm} \psi_d^j \psi_d^k \psi_d^l \psi_d^m + T_{dddp}^{jklm} \psi_d^j \psi_d^k \psi_d^l \psi_p^m \end{aligned} \tag{14}$$

and

$$\begin{aligned} U = {} & U^0 + U_d^j \psi_d^j + U_f^j \psi_f^j + U_{dd}^{jk} \psi_d^j \psi_d^k + U_{df}^{jk} \psi_f^j \psi_f^k + U_{ff}^{jk} \psi_f^j \psi_f^k \\ & + U_{ddd}^{jkl} \psi_d^j \psi_d^k \psi_d^l + U_{ddf}^{jkl} \psi_d^j \psi_d^k \psi_f^l + U_{dddd}^{jklm} \psi_d^j \psi_d^k \psi_d^l \psi_d^m + U_{dddf}^{jklm} \psi_d^j \psi_d^k \psi_d^l \psi_f^m . \end{aligned} \tag{15}$$

The coefficients T^0, T^j_d, T^j_p, T^{jk}_{dd}, T^{jk}_{dp}, T^{jk}_{pp}, T^{jkl}_{ddd}, T^{jkl}_{ddp}, T^{jklm}_{dddd}, T^{jklm}_{dddp} and U^0, U^j_d, U^j_f, U^{jk}_{dd}, U^{jk}_{df}, U^{jkl}_{ff}, U^{jkl}_{ddd}, U^{jkl}_{ddf}, U^{jklm}_{dddd}, U^{jklm}_{dddf} are functions of $\mathbf{d}^0$, $\mathbf{f}^{*0}$, $\mathbf{p}^{*0}$, $\mathbf{d}^j$, $\mathbf{f}^{*j}$ and $\mathbf{p}^{*j}$. If the rotations are expressed in terms of quaternions, the terms T^{jklm}_{dddd} and U^{jklm}_{dddd} are null.

It should be pointed out that the expressions (14) and (15) involve no additional assumptions if the Euler parameters have been used as a parameterization of the rotations. In fact, given the purely polynomial nature of the rotational operators $\mathbf{R}$ and $\boldsymbol{\Gamma}$ in terms of quaternions, the kinetic and the potential energies are simple quartic expression in the displacements, forces and momenta. On the contrary, if other parameterizations have been adopted such as the conformal rotation vector, the modal expansions involve infinite series, which must then be truncated. This is equivalent to a moderate rotation assumption, and the finite rotation case would be recovered only as the number of terms in the series increases to infinity. The quartic expressions of the kinetic and strain energies given in (14) and (15) imply the dropping of terms of fourth and higher degree.

Perturbation Modes

Modal reduction methods are based on the implicit assumption that the response of a system can be represented by the superposition of a small number of given modes. While it is well known that this is an effective procedure for linear analysis, there exists no proof of convergence or accuracy in the context of non-linear problems. Moreover, the choice of appropriate modal bases has been shown to be of crucial importance. In fact, it has been shown [6] that the use of natural vibration modes is not always suitable for an accurate and reliable description of non-linear systems. This result is not surprising, since natural modes characterize the linearized response of a system, i.e., the behavior of small perturbations about an equilibrium reference condition.

Indeed, the success of a modal analysis lies in the ability of identifying a suitable modal base effectively representing the response of the system. In the limit case, if one could know the real solution to the problem, one single mode represented by the solution itself would suffice. The importance of identifying appropriate bases is clear, as demonstrated by the relevant research effort developed over the years by a number of authors.

The concept of perturbation modes seems to present an effective choice of modal bases for non-linear problems. The use of perturbation modes in finite element formulations has been introduced by Thompson and Walker [8] in 1968. In their approach, the solution is expanded in terms of the perturbation modes in the tradition of perturbation theory, and the convergence characteristics were poor. On the contrary, Noor has shown in his classical papers [9, 10, 11, 12, 13] how perturbation modes provide excellent results even for highly non-linear problems, when employed as bases for modal reduction methods as described in the previous section.

In this work, space-time perturbation modes have been used for the non-linear dynamic modal analysis. These modes extract information about the non-linear kinematic and dynamic behavior of the system from higher order derivatives of the strain and kinetic energies within the framework of the finite element in space-time methodology.

The incremental matrix form of the space-time finite element equations derived from the energy principle (3) was developed in a previous sections and it is here repeated for convenience.

$$\mathbf{T}(\mathbf{d})\Delta\mathbf{d} = \mathbf{Q} - \mathbf{R}(\mathbf{d}), \tag{16}$$

where $\mathbf{d}$ is the vector of displacement nodal values, while $\mathbf{T}$ is the tangent matrix, $\mathbf{R}$ is the residual vector and $\mathbf{Q}$ are equivalent nodal forces.

At equilibrium $\Delta\mathbf{d} = 0$ and thus

$$\mathbf{R}(\mathbf{d}) = \mathbf{Q}. \tag{17}$$

The applied load can now be thought as a function of a scalar parameter λ

$$\mathbf{R}(\mathbf{d}(\lambda)) - \lambda\mathbf{Q} = 0. \tag{18}$$

If equation (18) holds, all its derivatives with respect to the parameter are null. Taking the first derivative of equation (18) with respect to λ, the first perturbation mode $\mathbf{d}^{(1)}$ is obtained as

$$\frac{\partial R_i}{\partial \mathrm{d}_j}\,\mathrm{d}_j^{(1)} = Q_i, \tag{19}$$

which is clearly the solution to the linearized problem. Note that $(\partial R_i)/(\partial\,\mathrm{d}_j) = T_{ij}$. Higher order perturbation modes can be readily obtained by means of the following recursive relations, which can be obtained as successive derivatives of equation (18)

$$T_{ij}\,\mathrm{d}_j^{(2)} = -\frac{\partial^2 R_i}{\partial\,\mathrm{d}_j \partial\,\mathrm{d}_k}\,\mathrm{d}_j^{(1)}\,\mathrm{d}_k^{(1)}, \tag{20}$$

$$T_{ij}\,\mathrm{d}_j^{(3)} = -3\frac{\partial^2 R_i}{\partial\,\mathrm{d}_j \partial\,\mathrm{d}_k}\,\mathrm{d}_j^{(1)}\,\mathrm{d}_k^{(2)} - \frac{\partial^3 R_i}{\partial\,\mathrm{d}_j \partial\,\mathrm{d}_k \partial\,\mathrm{d}_l}\,\mathrm{d}_j^{(1)}\,\mathrm{d}_k^{(1)}\,\mathrm{d}_l^{(1)}. \tag{21}$$

Similarly for the higher order perturbation modes.

It should be pointed out that the extraction of all the modes always necessitates of the factorization of the same tangent matrix. Thus the computational cost associated with the evaluation of a new perturbation mode is only related with the formation of the right hand side of its matrix equation. The perturbation modes for forces and momenta can be recovered a posteriori at the element level exactly as in the full finite element procedure.

Numerical Studies

This section will focus on simple 2-D beam problems in order to demonstrate the validity of the present approach by means of numerical experimentation. The results obtained using the space-time perturbation modes have been compared with those produced by the full finite element analysis, i.e., when all the degrees of freedom are considered, which is regarded here as the "reference" solution. It should be pointed out that this choice is pertinent for the purpose of comparison, since in this way the full finite element model, the modal basis and the modal reduction are all based on the same identical finite element discretization of the physical problem, the only difference lying in the space-time representation of the solution. Moreover, analytical results, when available in the literature, are also presented.

In the first example here discussed, a beam is clamped at both ends and it is subject to a sinusoidal-in-time spatially constant load $f_e = f_e^0 \cos(2\pi F_e \cdot t)$. The load acts in a fixed inertial direction normal to the axis of the undeformed beam and its frequency is close to the first natural frequency of the beam. The physical properties of the beam are the following: length $L = 3.15$ m; axial stiffness $EA = 7.30 \cdot 10^6\ N$; shearing stiffness $GA\alpha = 2.34 \cdot 10^6\ N$; bending stiffness $EI = 60.83\ N\ \mathrm{m}^2$; mass per unit span $m = 0.27$ kg/m, being E the

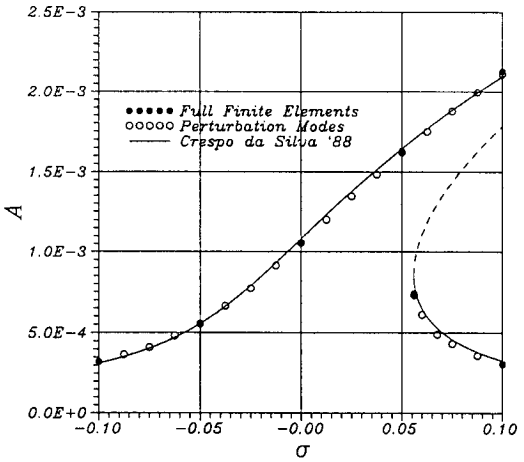

Fig. 1. Clamped-clamped beam problem: non-dimensional amplitude of the response at the mid point of the beam for an excitation frequency close to the first natural frequency.

Young's modulus, G the shear modulus, A the cross-sectional area, α a knock-down factor depending on the cross-sectional shape and I the section moment of inertia. The periodicity condition is imposed as previously explained. Three four-noded elements in space and 60 two-noded elements in time have been used for this problem. Due to symmetric boundary conditions in space, only half of the beam was modeled. The modal basis consists of the first two space-time perturbation modes computed in correspondence of the natural frequency and at the undeformed reference configuration.

In Figure 1 the non-dimensional mid point amplitude of the response of the beam in correspondence to an applied external load of intensity $f_e^0 L^3/EI = 0.024$ is plotted as a function of the detuning $\sigma = F_e/F_1 - 1$, being F_1 the first natural frequency of the beam. The non-linear effects due to midplane-stretching are evident. For the purpose of comparison, the same response has been computed using the full finite element formulation of (3) and the analytical results obtained in [14, 15]. A very good correlation with the analytical results can be observed in Figure 1. Furthermore, the correlation with the full finite element solution is almost exact.

It seems interesting to emphasize that a very small number of modes has been used. Moreover, it should be pointed out that the modal basis computed at the natural frequency F_1 has been used over the entire range of the detuning $-0.10 \leq \sigma \leq 0.10$. This allows a drastic reduction in computational cost with respect to the full finite element solution. Letting the computational effort implied by a full finite element response analysis performed at a given load level and response period be equal to one, the cost of determining a modal basis of two modes has been found for this problem to be equal to 0.36, and 0.56 for the calculation of a four mode basis. Moreover, the solution phase has an extremely low cost, since the space-time modal reduction scheme results in a set of algebraic equations in the generalized modal coordinates. In particular, the cost needed by the reduced technique for performing a response analysis at a given load level and response period has been found to be 0.1 using two modes, and 0.16 using four modes.

Figure 2 shows the response of the system when shearing deformations are taken into account. The shear parameter $s^2 = EI/(GA\alpha \cdot L^2)$ is defined to indicate a measure of the shear effects. The detuning is measured with respect to the first natural frequency of the structure,

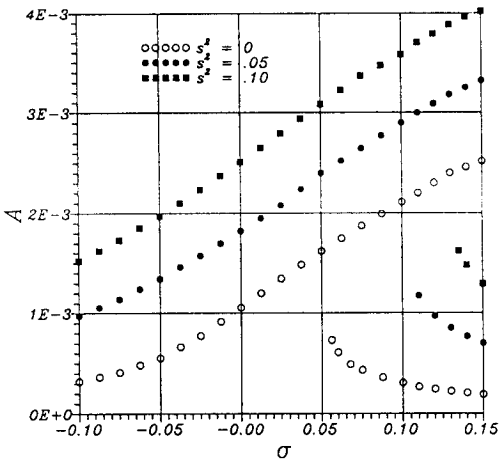

Fig. 2. Clamped-clamped beam problem: shear effects.

which is $F_1\sqrt{EJ}/L^2\sqrt{m} = 4.47\cdot 10^{-2}$ at $s^2 = 0$, $F_1\sqrt{EJ}/L^2\sqrt{m} = 8.38\cdot 10^{-2}$ at $s^2 = 0.05$ and $F_1\sqrt{EJ}/L^2\sqrt{m} = 1.10\cdot 10^{-1}$ at $s^2 = 0.10$. This example has been introduced with the purpose of stressing the importance of finite element based modal reduction techniques. Indeed, in the context of finite elements, shear effects or particularly complex geometric configurations do not imply any additional difficulty. On the contrary, the analytical methods proposed in the literature, although invaluable in order to understand the basic behavior of a system, often involve simplifying hypothesis, such as for example constant properties along the beam axis, absence of shear effects, initially straight configuration, moderate rotations, just to mention a few.

As a further example, a beam clamped at one end and free at the other is here discussed. The beam is denoted by the same physical properties of the previous example and it is once again subjected to a sinusoidal-in-time spatially constant load of intensity $f_e^0 L^3/EI = 0.03$. The finite element model consists of three four-noded elements in space and 60 two-noded elements in time, while the periodicity condition has been imposed in the usual way.

The case in which the frequency of the forcing function is close to the second natural frequency, is discussed first. Figure 3 shows the non-dimensional tip amplitude of the beam as a function of the detuning $\sigma = F_e/F_2 - 1$. Here again, the correlation with the results obtained using the full finite elements and the analytical results [16, 17] is very good. The modal basis consisting of the first two space-time perturbation modes was computed at the detuning value of $\sigma = 0$ about the undeformed reference configuration, and it was used over the entire detuning range $-0.010 \leq \sigma \leq 0.010$. In order to illustrate the nature of the perturbation modes, the shape in space-time of some of the components of the first three modes are depicted in Figure 4 for the clamped beam problem. When the modes are computed about the undeformed configuration, the even numbered modes are characterized by pure shortening and no deflection, while the odd numbered modes are denoted by pure deflection and zero shortening.

When the load frequency is close to the first natural frequency of the beam, the amplitude of the response is relevant for values of the detuning $-0.005 \leq \sigma \leq 0.010$. As it has been shown in the previous sections, the perturbation modes can be calculated without resorting to the moderate rotation assumption for any parameterization of the finite rotation; however

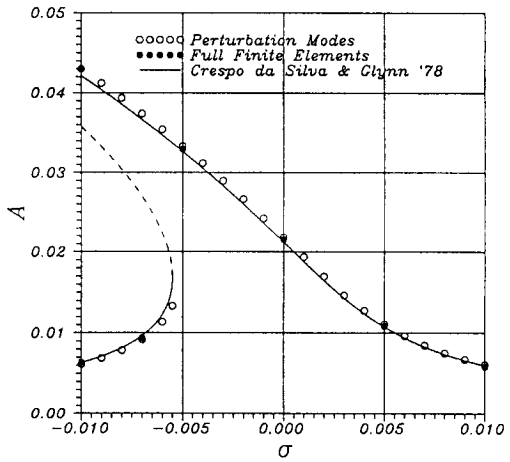

Fig. 3. Clamped-free beam problem: non-dimensional amplitude of the response at the free end of the beam for an excitation frequency close to the second natural frequency.

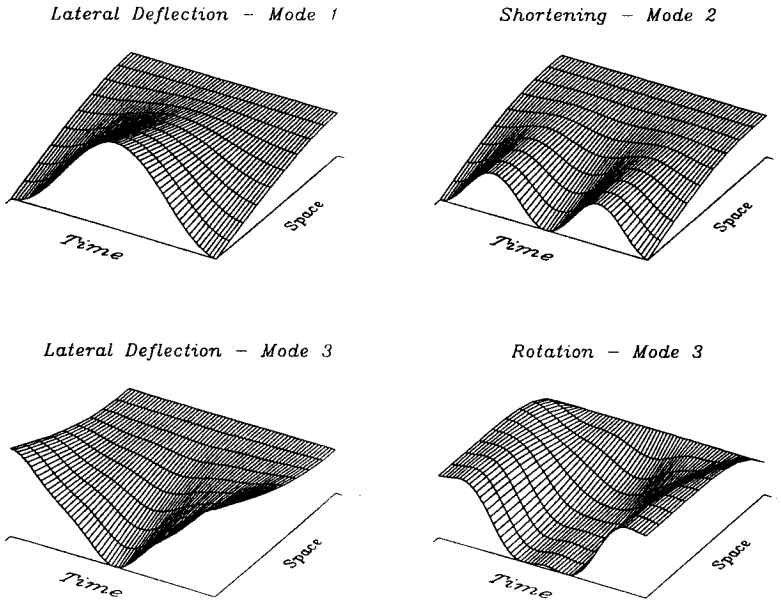

Fig. 4. Clamped-free beam problem: shape of the perturbation modes in space-time.

the coefficients of the reduction process must be truncated if the conformal rotation vector is adopted, thus implying a moderate rotation approximation. The effects of this assumption are apparent in Figure 5. A modal basis comprising four modes has been computed at the maximum deflection configuration at $\sigma = 0.01$. If the quaternions are used, the finite rotations are rigorously treated and the response computed with the modal reduction process is in perfect agreement with the result obtained from the full finite element model. On the other hand, if the conformal rotation vector is used, the moderate rotation assumption is manifest at the highest values of the tip deflection. In particular, it is interesting to note that, although the modal basis has been computed at the reference configuration corresponding to the large deflection at $\sigma = 0.010$, the modal solution at this detuning does not render the same solution,

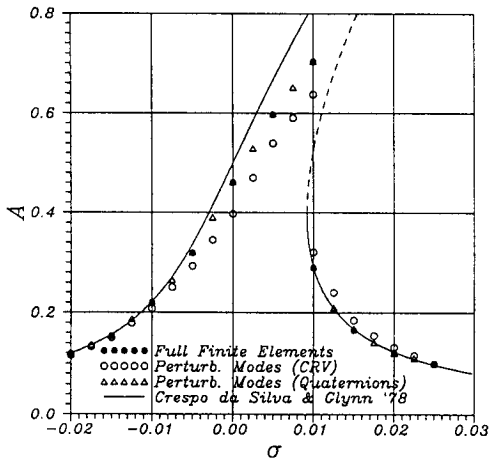

Fig. 5. Clamped-free beam problem: non-dimensional amplitude of the response at the free end of the beam for an excitation frequency close to the first natural frequency.

because of the moderate rotation approximation. It should be remarked the good correlation between the full finite element model and the analytical solution [16] based on a perturbation analysis expanded to the cubic order of the non-linearities. Excellent agreement is found for deflections up to 40% of the maximum deflection considered.

It should be noted that the results relative to this last example are possible only because space-time perturbation modes have been used. In this way, certain difficulties experienced with static perturbations modes, i.e., modes computed from the discrete static equilibrium equations, when used for dynamic problems, are removed. In fact it has been shown in [6] that when a static modal basis is computed about a predeformed configuration, it performs well only when the dynamic response is in the same direction of the predeformation. On the contrary, it performs very poorly when the dynamic response is in the opposite direction of the predeformation. Hence, modes about a statically deformed configuration should be avoided if the dynamic response involves complete reversal, as it is the case for this particular problem. However, since space-time perturbation modes extract information about the global kinematic-dynamic behavior of the structure, the modal basis brings information about the non-linear deflection of the beam on both sides of the straight configuration.

Conclusions

In this work a finite element based modal reduction methodology for non-linear dynamics has been presented. A mixed three-field variational formulation has been used to develop a space-time finite element discretization. The assumed modal basis was chosen to be a set of perturbation modes obtained directly from this space-time formulation. This generalization of the perturbation mode concept, which is naturally obtained in the context of finite elements in space-time, generates modes which extract information about the non-linearities of the system through the higher order derivatives of the kinetic and potential energies. Furthermore, the reduced equations obtained in this way are of a purely algebraic nature.

Two different parameterizations of finite rotations have been investigated, namely the conformal rotation vector and the quaternions. While the first is computationally convenient since it is not redundant, only the latter can treat arbitrarily large rotations in the context of modal

reduction techniques. An effective formulation for dealing with the unitary quaternion condition has been discussed, avoiding the penalty method in favor of the augmented Lagrangian approach.

With reference to the beam example problems here addressed, it has been shown how the results obtained with the proposed methodology are in excellent agreement with those computed with the full finite elements based on the same variational principle. It has been pointed out that the computational effort associated with the modal reduction is extremely low when compared with the full finite elements.

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