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The Vector Parameterization of Motion

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Abstract. This paper presents a vector parameterization of motion that generalizes the vector parameterization of rotation. The Plücker coordinates of an arbitrary material line of a rigid body subjected to a screw motion are shown to transform by the action of a motion tensor. The proposed vector parameterization completely describes an arbitrary motion by means of two vectors that constitute an eigenvector of the motion tensor associated with its positive unit eigenvalue. The first vector is conveniently selected as the vector parameterization of rotation, and the second is related to the displacement of a point of the rigid body. A complete description of motion is presented in terms of a generic vector parameterization. Relevant formulæ for specific parameterizations of this class can then be easily obtained. More details are given for three parameterizations that present desirable properties: the Euler, Cayley–Gibbs–Rodrigues, and Wiener–Milenkovic motion parameters. Applications to the dynamic analysis of a rigid body are presented as an illustration.

Keywords: Motion tensor, operations, functions and motions.

1. Introduction

The effective description of rotation has led over the years to the development of numerous techniques, presenting various properties and advantages. Reviews of these parameterization techniques may be found in [1–6]. Whether originating from geometric, algebraic, or matrix approaches, parameterizations of rotation are most naturally categorized into two classes: *vector* and *non-vector* parameterizations. The former refers to parameterization in which a set of parameters (sometimes called rotational ‘quasi-coordinates’) define a geometric vector, whereas the latter cannot be cast in the form of a vector.

Euler’s theorem on rotation states that an arbitrary motion of a rigid body that leaves one of its point fixed can be represented by a single rotation of magnitude ϕ about a unit vector $\bar{n}$. It is readily shown that the associated *rotation tensor* R possesses a positive unit eigenvalue and the corresponding eigenvector is $\bar{n}$. Bauchau and Trainelli [7] introduced the *vector parameterization of rotation* that consists of a minimal set of parameters defining the components of a *rotation parameter vector*, $\underline{p} = p(\phi)\,\bar{n}$. Hence, all parameter vectors are eigenvectors of the rotation tensor associated with its positive unit eigenvalue, with magnitude $p(\phi) = \|\underline{p}\|$. Clearly, a specific vector parameterization is completely defined by the choice of the *generating function* $p(\phi)$. The unit vector $\bar{n}$ is the axis about which the rotation takes place, but it does not determine its magnitude. The vector parameterization remedies this situation by combining the relevant information into the rotation parameter vector $\underline{p}$. The exponential map of rotation, the Euler, Cayley–Gibbs–Rodrigues, and Wiener–Milenkovic parameterizations all are special cases of the vector parameterization corresponding to specific choices of the generating function. This generalized parameterization sheds additional light on the fundamental properties of these techniques, pointing out the similarities in their formal structure and showing their inter-relationships.

This paper presents a *vector parameterization of motion* that generalizes the vector parameterization of rotation. Mozzi–Chasles' theorem states that an arbitrary motion of a rigid body can be represented by a screw motion. The axis of the screw is called the *Mozzi–Chasles axis*, denoted $\underline{\mathcal{M}}$, and its Plücker coordinates [8] can be readily evaluated [9, 10]. The Plücker coordinates of an arbitrary material line of a rigid body subjected to a screw motion are shown to transform by the action of the *motion tensor*, and $\underline{\mathcal{M}}$ to be an eigenvalue of this tensor associated with its positive unit eigenvalue. The Mozzi–Chasles axis is the axis about which the screw motion takes place, but it does not determine the magnitude of the rotation and translation about and along this axis. The vector parameterization of motion remedies this situation by combining all the relevant information into two vectors, $\underline{q}$ and $\underline{p}$, that constitute an eigenvector of the motion tensor associated with its positive unit eigenvalue. $\underline{p}$ is conveniently selected to be the vector parameterization of rotation, and $\underline{q}$ is related to the displacement of a point of the rigid body.

A complete description of motion is presented for a generic vector parameterization. Relevant formulæ for specific parameterizations of this class can then be easily obtained. It is even possible to devise new parameterization techniques subjected to given requirements. Specific expressions are given for three parameterizations that present desirable properties: the Euler, Cayley–Gibbs–Rodrigues, and Wiener–Milenkovic motion parameters. The term 'motion parameters' will be used here to make a distinction between the classical rotation parameters (e.g. Euler parameters) and the proposed motion parameters (e.g. Euler motion parameters). Applications to dynamic problems are straightforward by extending the classical concepts of virtual motion and velocity to the proposed parameterization of motion.

The proposed vector parameterization of motion is closely related to other approaches presented in the literature. For instance, the proposed Euler motion parameters are closely related to the dual-number quaternion algebra technique used in kinematics [11–13]. Dual numbers, vectors and matrices [14] have also received considerable attention in kinematics [8, 10, 15] and dynamics [16]. Many of these studies have shown that the most efficient and elegant implementations of dual-number techniques are based on general screw theory with the screw expressed by means of Plücker coordinates; this approach is also the starting point of this work. A systematic, coordinate-free exposition of the different algebraic operations in the set of infinitesimal displacements (screws) and their relation with finite displacement was developed by Chevallier [17]. Finally, Borri et al. [18] have recently proposed parameterizations based the exponential map of motion and on Cayley's parameterization. These two specific parameterizations are special cases of the proposed vector parameterization of motion.

This paper is organized in the following manner. In Section 2, the motion tensor is introduced as the tensor that transforms the Plücker coordinates of a line of a rigid body undergoing a general screw motion. The motion tensor is also shown to perform change of frame operations for kinematic and co-kinematic quantities. Next, time derivatives of finite motion operations are discussed in Section 3. Virtual motions and velocities are also introduced. The Euler motion parameters are presented in Section 4, and the basic formulæ for manipulating finite motions are given in terms of these parameters. The vector parameterization of motion is introduced in Section 5, and specific choices of the generating function are proposed in Section 6. In Section 7, two parameterizations of motion are presented in detail. The paper concludes with the application of one of the proposed parameterizations to the dynamic analysis of a rigid body.

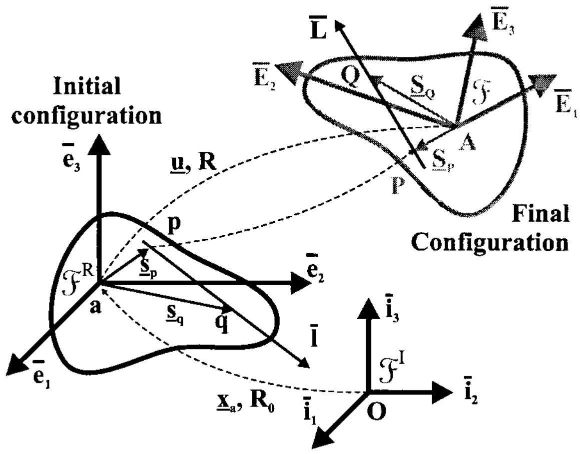

Figure 1. A line of a rigid body in the initial and final configurations.

2. The Motion Tensor

2.1. Transformation of a Line of a Rigid Body

Figure 1 depicts a rigid body in its reference configuration as defined by frame $\mathcal{F}^R = (\mathbf{a}, \mathcal{B}^R)$, where $\mathbf{a}$ is a reference point on the body and $\mathcal{B}^R = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$ the orthonormal basis that determines its orientation. Let $\underline{s}_p$ and $\underline{s}_q$ be the relative position vectors of point $\mathbf{p}$ and $\mathbf{q}$ of the body, respectively, with respect to point $\mathbf{a}$. The rigid body now undergoes an arbitrary motion that brings it to a final configuration defined by a frame $\mathcal{F} = (\mathbf{A}, \mathcal{B})$, where point $\mathbf{A}$ is the reference point of the body and $\mathcal{B} = (\bar{E}_1, \bar{E}_2, \bar{E}_3)$ a body attached orthonormal basis. The displacement vector of point $\mathbf{a}$ is denoted $\underline{u}$. The relative position vectors of points $\mathbf{P}$ and $\mathbf{Q}$ with respect to point $\mathbf{A}$ are denoted $\underline{S}_P$ and $\underline{S}_Q$, respectively. Finally, let R be the rotation tensor that brings basis $\mathcal{B}^R$ to $\mathcal{B}$. All vectors and tensors are measured in the inertial frame $\mathcal{F}^I = (\mathbf{O}, \mathcal{B}^I)$, where $\mathcal{B}^I = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ is an orthonormal basis.

Consider a line joining two points of a rigid body, such as line $\mathbf{pq}$ in Figure 1. In the initial configuration, the line is defined by one of its points, say $\mathbf{p}$, and its orientation

$$\bar{l} = \frac{\underline{s}_q - \underline{s}_p}{\|\underline{s}_q - \underline{s}_p\|}. \tag{1}$$

The Plücker coordinates [8] of this line are

$$\underline{\mathcal{Q}}^* = \begin{bmatrix} \tilde{s}_p \bar{l} \\ \bar{l} \end{bmatrix} = \begin{bmatrix} \underline{q} \\ \bar{l} \end{bmatrix}. \tag{2}$$

The skew-symmetric matrix formed with the components of a vector $\underline{s}$ is denoted $\tilde{s}$. Note that the coordinates[1] were evaluated with respect to point $\mathbf{a}$. In the final configuration, the orientation of the line becomes

$$\bar{L} = \frac{\underline{S}_Q - \underline{S}_P}{\|\underline{S}_Q - \underline{S}_P\|}, \tag{3}$$

[1] Some authors define the Plücker coordinates as $\underline{\mathcal{Q}}^{*T} = \lfloor \bar{l}^T, \underline{q}^T \rfloor$, reversing the order of the two vectors.

and its Plücker coordinates with respect to point **a** are

$$\underline{\mathcal{Q}} = \begin{bmatrix} (\tilde{u} + \tilde{S}_P)\, \bar{L} \\ \bar{L} \end{bmatrix} = \begin{bmatrix} \underline{Q} \\ \bar{L} \end{bmatrix}. \tag{4}$$

Since the body is rigid, $\underline{S}_P = R\,\underline{s}_p$ and $\underline{S}_Q = R\,\underline{s}_q$; it then follows that $\bar{L} = R\,\bar{l}$, and hence

$$\underline{\mathcal{Q}} = \begin{bmatrix} \tilde{u}\,R\,\bar{l} + R\,\tilde{s}_p\,\bar{l} \\ R\,\bar{l} \end{bmatrix} = \begin{bmatrix} R & \tilde{u}\,R \\ 0 & R \end{bmatrix} \begin{bmatrix} \tilde{s}_p\,\bar{l} \\ \bar{l} \end{bmatrix}. \tag{5}$$

The *motion tensor* is defined as

$$\mathcal{C} = \begin{bmatrix} R & \tilde{u}\,R \\ 0 & R \end{bmatrix}, \tag{6}$$

and Equation (5) can now be written in a compact form as

$$\underline{\mathcal{Q}} = \mathcal{C}\,\underline{\mathcal{Q}}^*. \tag{7}$$

Clearly, the motion tensor is the operator that transforms the Plücker coordinates of an arbitrary line of the rigid body in the initial configuration to their counterparts in the final configuration [8, 10, 15].

In general, an arbitrary line of a rigid body is different in the initial and final configurations. The following question can then be asked: does there exist a line of the rigid body that is identical in the initial and final configurations? If such line exists, its Plücker coordinates in the initial and final configurations are identical, i.e. $\underline{\mathcal{Q}} = \underline{\mathcal{Q}}^*$. This first implies $\bar{l} = R\,\bar{l}$, i.e. $\bar{l}$ must be the eigenvector of the rotation tensor corresponding to a unit eigenvalue. It is well known that this eigenvector is the axis of the rotation, $\bar{n}$, and hence $\bar{l} = \bar{n}$. Next, it also implies $R\,\underline{q} + \tilde{u}\,R\,\bar{l} = \underline{q}$, or $(R - I)\,\underline{q} = \tilde{n}\,\underline{u}$. In view of [7, equation (81)], $R - I = GG - G^T G = (G - G^T)G = 2\sin\phi/2\,\tilde{n}\,G$ and this equation becomes

$$\tilde{n}\left(2\sin\frac{\phi}{2}\,G\,\underline{q} - \underline{u}\right) = 0. \tag{8}$$

The non-trivial solution of this homogeneous problem is in the null space of $\tilde{n}$, which implies $2\sin\phi/2\,G\,\underline{q} = \underline{u} + \beta\,\bar{n}$, where β is an arbitrary constant. Since $\underline{q}$ is part of the Plücker coordinates, it must be orthogonal to $\bar{n}$, and hence, $\beta = \bar{n}^T\,\underline{u}$. It follows that $2\sin\phi/2\,G\,\underline{q} = (I - \bar{n}\,\bar{n}^T)\,\underline{u} = -\tilde{n}\,\tilde{n}\,\underline{u}$, and finally

$$\underline{q} = -\frac{G^T\,\tilde{n}\,\tilde{n}}{2\sin\phi/2}\,\underline{u}. \tag{9}$$

In summary, the Plücker coordinates of the line of the rigid body that is identical in the initial and final configurations, are $\underline{q}$, given by Equation (9), and $\bar{l} = \bar{n}$. However, these coordinates are those of the Mozzi–Chasles axis associated with the motion of the body [9, 10]. Hence, the Mozzi–Chasles axis is the line of the rigid body that is identical in the initial and final configurations. This can be written as $\underline{\mathcal{Q}}_{MC} = \mathcal{C}\,\underline{\mathcal{Q}}_{MC}$: the Mozzi–Chasles axis is an eigenvector of the motion tensor corresponding to a unit eigenvalue.

2.2. Frame Change for Kinematic Quantities

Let the final configuration of the rigid body depicted in fig. 1 be time dependent whereas the reference configuration is that of the body at a fixed, initial time. Consider now two vectors associated with the rigid body: the velocity vector at point **A**, $\underline{v}_A$, a bound vector, and the angular velocity vector of the body, $\underline{\omega}_A$, a free vector. The components of these two vectors measured in basis $\mathcal{B}$ are denoted $\underline{v}_A^*$ and $\underline{\omega}_A^*$, respectively, where the subscript on the latter symbol is, of course, superfluous since the angular velocity is identical for all points of the body. The following velocity vector is now defined:

$$\underline{\mathcal{V}}^* = \begin{vmatrix} \underline{v}_A^* \\ \underline{\omega}_A^* \end{vmatrix}. \tag{10}$$

Since the motion tensor was shown in the previous section to transform the Plücker coordinates of a line from one frame to the other, it is interesting to consider the following transformation:

$$\underline{\mathcal{V}} = \begin{vmatrix} \underline{v}_a \\ \underline{\omega}_a \end{vmatrix} = \mathcal{C}\, \underline{\mathcal{V}}^*. \tag{11}$$

To understand the physical meaning of this transformation, the physical meaning of the velocity vectors $\underline{v}_a$ and $\underline{\omega}_a$ must first be identified. It is clear that $\underline{v}_A = R\, \underline{v}_A^*$ and $\underline{\omega}_A = R\, \underline{\omega}_A^*$ are the components of vectors $\underline{v}_A$ and $\underline{\omega}_A$, respectively, measured in basis $\mathcal{B}^R$. This corresponds to a *change of basis* operation that establishes the relationship between the components of vectors in two orthonormal bases. Next, since $\underline{v}_a = R\, \underline{v}_A^* + \tilde{u}\, R\, \underline{\omega}_A^* = \underline{v}_A - \tilde{\omega}_A\, \underline{u}$, the velocity vector $\underline{v}_a$ is that of the point of the rigid body which instantaneously coincides with the origin of the reference frame, point **a**. Of course, $\underline{\omega}_A$ can also be interpreted as the angular velocity vector of the same point, since the angular velocity vector is the same for all points of a rigid body. Hence, this second operation corresponds to a *change of reference point* that establishes the relationship between the velocities of two different points of the rigid body.

In summary, the operation described by Equation (11) corresponds to a *change of frame* that combines a change of basis and a change of reference point [18]. The motion tensor can be factorized in the following manner

$$\mathcal{C} = \begin{bmatrix} I & \tilde{u} \\ 0 & I \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} = \mathcal{T}\, \mathcal{R}. \tag{12}$$

This factorized form clearly underlines the double effect of a frame change; it consists of two operation: first a change of basis operation characterized by the *rotation operator* $\mathcal{R}$, then a change of reference point operation characterized by the *translation operator* $\mathcal{T}$.

This change of frame operation can be inverted to yield

$$\underline{\mathcal{V}}^* = \mathcal{C}^{-1}\, \underline{\mathcal{V}}, \tag{13}$$

where the inverse of the motion tensor is

$$\mathcal{C}^{-1} = \begin{bmatrix} R^T & R^T\, \tilde{u}^T \\ 0 & R^T \end{bmatrix}. \tag{14}$$

2.3. Frame Change for Co-Kinematic Quantities

A similar study can be made concerning two other vectors associated with the rigid body: the force vector acting at a point **A**, $\underline{f}_A$, a bound vector, and the moment acting on the body, $\underline{m}_A$, a free vector. The components of these two vectors measured in basis $\mathcal{B}$ are denoted $\underline{f}^*_A$ and $\underline{m}^*_A$, respectively, where the subscript on the former symbol is, of course, superfluous since the force vector is the same at all points of the rigid body. The following force vector is defined:

$$\underline{\mathcal{F}}^* = \left| \begin{matrix} \underline{f}^*_A \\ \underline{m}^*_A \end{matrix} \right| . \tag{15}$$

Consider now the effect of the following transformation:

$$\underline{\mathcal{F}} = \left| \begin{matrix} \underline{f}_a \\ \underline{m}_a \end{matrix} \right| = \mathcal{C}^{-T} \, \underline{\mathcal{F}}^*. \tag{16}$$

Here again, it is clear that $\underline{f}_A = R\,\underline{f}^*_A$ and $\underline{m}_A = R\,\underline{m}^*_A$ are the components of the force and moment vectors, respectively, in basis $\mathcal{B}^R$. This corresponds to a change of basis operation. Next, since $\underline{m}_a = R\,\underline{m}^*_A + \tilde{u}\,R\,\underline{f}^*_A = \underline{m}_A + \tilde{u}\,\underline{f}_A$, the moment vector $\underline{m}_a$ is the applied moment computed with respect to the point of the rigid body that instantaneously coincides with point **a**. Of course, $\underline{f}_a$ can also be interpreted as the force applied on the rigid body at the same point, since this force is the same at all points of the rigid body.

In summary, the operation described by Equation (16) is a change of frame operation that combines a change of basis and a change of reference point [18]. This change of frame operation can be inverted to yield

$$\underline{\mathcal{F}}^* = \mathcal{C}^T \, \underline{\mathcal{F}}. \tag{17}$$

It is important to note that the components kinematic and co-kinematic quantities transform differently under a frame change operation, as indicated in Equations (11) and (16), respectively. Both transformations, however, are based on the motion tensor which appears to be a fundamental quantity associated with frame changes. It is clear that $\mathcal{C}^{-T} = \mathcal{T}^{-T}\,\mathcal{R}$. This factorized form underlines again the double effect of a frame change for co-kinematic quantities; it consists of two operations: first, a rotation operation using the rotation operator $\mathcal{R}$, then a translation operation using the reference point translation operator $\mathcal{T}^{-T}$.

2.4. Properties of the Motion Tensor

The eigenvalues of the motion tensor are easily computed from Equation (12). Indeed

$$\det(\mathcal{C}) = \det(\mathcal{T})\,\det(\mathcal{R}) = \det(\mathcal{T})\,(\det(R))^2. \tag{18}$$

Since $\det(\mathcal{T}) = 1$, it follows that $\det(\mathcal{C}) = (\det(R))^2$. Hence, the eigenvalues of the motion tensor are identical to those of the rotation tensor, but each with a multiplicity of two. However, the motion tensor, unlike the rotation tensor, is not an orthogonal tensor. An eigenvector $\underline{\mathcal{N}}$ associated with the unit eigenvalue is denoted

$$\underline{\mathcal{N}} = \left| \begin{matrix} \underline{q} \\ \underline{p} \end{matrix} \right| . \tag{19}$$

If $\underline{\mathcal{N}}$ is the desired eigenvector, if must satisfy the following relationship

$$\begin{bmatrix} R-I & \tilde{u}\,R \\ 0 & R-I \end{bmatrix} \begin{vmatrix} \underline{q} \\ \underline{p} \end{vmatrix} = 0. \tag{20}$$

The bottom three equations imply $(R-I)\,\underline{p} = 0$. Since $R-I$ is a singular matrix, $\underline{p}$ must be in the null space of $R-I$, $\bar{n}$. Hence, a valid choice of $\underline{p}$ is the vector parameterization of R. Next, the top three equations imply $(R-I)\,\underline{q} - \tilde{p}\,\underline{u} = \overline{0}$. Introducing equation (18) from [7] then leads to $\tilde{p}\,(\nu G\,\underline{q} - \underline{u}) = 0$. Since $\tilde{p}$ is a singular matrix, the homogeneous system admits a non trivial solution $\nu G\,\underline{q} = \underline{u} + \mu\,\underline{p}$, where μ is an arbitrary coefficient. It is convenient to select this coefficient as $\mu = \alpha(\underline{p}^T\underline{u})/p^2$, where α is, again, an arbitrary coefficient. It then follows that

$$\underline{q} = \frac{1}{\nu}\left[G^T + \alpha\frac{\underline{p}\,\underline{p}^T}{p^2}\right]\underline{u} = D\,\underline{u}, \tag{21}$$

where operator D writes

$$D = \frac{1+\alpha}{\nu} - \frac{1}{2}\,\tilde{p} + \frac{1}{p^2}\left(\frac{1+\alpha}{\nu} - \frac{1}{\varepsilon}\right)\tilde{p}\tilde{p}. \tag{22}$$

The desired eigenvector now becomes

$$\underline{\mathcal{N}} = \begin{vmatrix} D\,\underline{u} \\ \underline{p} \end{vmatrix}. \tag{23}$$

Numerous solutions are possible, due to the presence of the arbitrary coefficient α; this stems from the fact that the eigenvalue $\lambda = +1$ of $\mathcal{C}$ has a multiplicity of two. Hence, any linear combination of two eigenvectors associated with this eigenvalue is still an eigenvector. If an additional constraint is applied, this coefficient can be evaluated. For instance, imposing the orthogonality condition $\underline{q}^T\,\underline{p} = 0$ implies $\alpha = -1$, and the resulting eigenvector then corresponds to the Plücker coordinates of the Mozzi–Chasles axis, as discussed in Section 2.1.

3. Derivatives of Finite Motion Operations

The derivatives of finite rotation operations lead to the concept of angular velocity vector. Dealing with derivatives of finite motion operations, both velocity and angular velocity vectors will emerge. Virtual changes in motion will also be investigated.

3.1. The Velocity Vector

Let frame $\mathcal{F} = (\mathbf{A}, \mathcal{B})$ depicted in Figure 1 be time dependent. Hence, the motion tensor $\mathcal{C}$ that brings reference frame $\mathcal{F}^R$ to frame $\mathcal{F}$ is also time dependent and Equation (7) then implies

$$\underline{\mathcal{Q}}(t) = \mathcal{C}(t)\,\underline{\mathcal{Q}}^*, \tag{24}$$

where t denotes time. Taking a time derivative of this equation leads to $\dot{\underline{\mathcal{Q}}} = \dot{\mathcal{C}}\,\underline{\mathcal{Q}}^*$, and eliminating $\underline{\mathcal{Q}}^*$ then yields

$$\dot{\underline{\mathcal{Q}}} = \dot{\mathcal{C}}\,\mathcal{C}^{-1}\,\underline{\mathcal{Q}}. \tag{25}$$

It is clear that expression $\dot{\mathcal{C}}\,\mathcal{C}^{-1}$ generalizes expression $\dot{R}R^T$ that leads to the definition of the angular velocity vector $\underline{\omega}$, such that $\tilde{\omega} = \dot{R}R^T$. This expression can be readily evaluated

$$\dot{\mathcal{C}}\,\mathcal{C}^{-1} = \begin{bmatrix} \dot{R} & \dot{\tilde{u}}\,R + \tilde{u}\,\dot{R} \\ 0 & \dot{R} \end{bmatrix} \begin{bmatrix} R^T & R^T\tilde{u}^T \\ 0 & R^T \end{bmatrix} = \begin{bmatrix} \tilde{\omega} & (\widetilde{\underline{\dot{u}} + \tilde{u}\,\underline{\omega}}) \\ 0 & \tilde{\omega} \end{bmatrix} = \begin{bmatrix} \tilde{\omega} & \tilde{v} \\ 0 & \tilde{\omega} \end{bmatrix}. \tag{26}$$

This gives rise to the velocity vector $\underline{v} = \underline{\dot{u}} + \tilde{u}\,\underline{\omega}$ and angular velocity vector $\underline{\omega} = \text{axial}(\dot{R}\,R^T)$. These quantities can be interpreted as the linear and angular velocities of the point of the rigid body that instantaneously coincides with the origin of the reference frame, point **a**, see Section 2.2. The following notation is now introduced:

$$\mathcal{W}(\underline{\mathcal{V}}) = \begin{bmatrix} \tilde{\omega} & \tilde{v} \\ 0 & \tilde{\omega} \end{bmatrix}, \tag{27}$$

and Equation (25) now becomes

$$\underline{\dot{\mathcal{Q}}} = \mathcal{W}(\underline{\mathcal{V}})\,\underline{\mathcal{Q}}. \tag{28}$$

The components of the derivative of the Plücker coordinates can also be measured in the material frame

$$\mathcal{C}^{-1}\,\underline{\dot{\mathcal{Q}}} = \mathcal{C}^{-1}\,\dot{\mathcal{C}}\,\underline{\mathcal{Q}}^*. \tag{29}$$

It is clear that expression $\mathcal{C}^{-1}\,\dot{\mathcal{C}}$ generalizes expression $R^T\dot{R}$ that leads to the components of the angular velocity vector in the material frame $\underline{\omega}^*$, such that $\tilde{\omega}^* = R^T\dot{R}$. It is readily found that

$$\mathcal{C}^{-1}\,\dot{\mathcal{C}} = \begin{bmatrix} R^T & R^T\tilde{u}^T \\ 0 & R^T \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{\tilde{u}}\,R + \tilde{u}\,\dot{R} \\ 0 & \dot{R} \end{bmatrix} = \begin{bmatrix} \tilde{\omega}^* & \widetilde{R^T\underline{\dot{u}}} \\ 0 & \tilde{\omega} \end{bmatrix} = \begin{bmatrix} \tilde{\omega}^* & \tilde{v}^* \\ 0 & \tilde{\omega}^* \end{bmatrix}. \tag{30}$$

This expression gives rise to the velocity vector $\underline{v}^* = R^T\underline{\dot{u}}$ and angular velocity vector $\underline{\omega}^* = \text{axial}(R^T\,\dot{R})$. These quantities are the components of the linear and angular velocities of point **A** of the rigid body, expressed in the material frame $\mathcal{F}$. Equation (29) now becomes

$$\mathcal{C}^{-1}\,\underline{\dot{\mathcal{Q}}} = \mathcal{W}(\underline{\mathcal{V}}^*)\,\underline{\mathcal{Q}}^*. \tag{31}$$

The above developments are summarized in the following relationships:

$$\dot{\mathcal{C}}\,\mathcal{C}^{-1} = \mathcal{W}(\underline{\mathcal{V}}); \quad \mathcal{C}^{-1}\,\dot{\mathcal{C}} = \mathcal{W}(\underline{\mathcal{V}}^*). \tag{32}$$

Similar developments lead to $\mathcal{C}\,\dot{\mathcal{C}}^{-1} = -\mathcal{W}(\underline{\mathcal{V}})$ and $\dot{\mathcal{C}}^{-1}\,\mathcal{C} = -\mathcal{W}(\underline{\mathcal{V}}^*)$.

3.2. The Virtual Velocity Vector

The virtual motion vector is introduced by analogy to the velocity vector by forming the following expression:

$$\begin{aligned} \delta\,\mathcal{C}\,\mathcal{C}^{-1} &= \begin{bmatrix} \delta R & \widetilde{\delta u}\,R + \tilde{u}\,\delta R \\ 0 & \delta R \end{bmatrix} \begin{bmatrix} R^T & R^T\tilde{u}^T \\ 0 & R^T \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{\delta\psi} & (\widetilde{\delta\underline{u} + \tilde{u}\,\delta\underline{\psi}}) \\ 0 & \widetilde{\delta\psi} \end{bmatrix} = \begin{bmatrix} \widetilde{\delta\psi} & \widetilde{\delta u} \\ 0 & \widetilde{\delta\psi} \end{bmatrix}. \end{aligned} \tag{33}$$

where $\underline{\delta\psi}$ is the virtual rotation vector defined as $\widetilde{\delta\psi} = \delta R R^T$, and $\underline{\delta u} = \delta\underline{u} + \tilde{u}\,\underline{\delta\psi}$. Note that there exist no vector $\underline{\psi}$ such that $\delta(\underline{\psi})$ is the virtual rotation vector. $\delta(\underline{u})$ represents the virtual displacement of point **A**, the frame reference point, whereas $\underline{\delta u} = \delta\underline{u} + \tilde{u}\,\underline{\delta\psi}$ represents a virtual change in the displacement of the material point of the rigid body that instantaneously coincides with point **a**. Of course, there exist no displacement vector, say $\underline{x}$, such $\delta(\underline{x}) = \delta\underline{u} + \tilde{u}\,\underline{\delta\psi}$. Hence, the notations $\underline{\delta u}$ and $\underline{\delta\psi}$ will be used to denote the virtual displacement and rotation vectors, respectively. By analogy to Equation (32), the following compact notation is adopted

$$\delta\mathcal{C}\,\mathcal{C}^{-1} = \mathcal{W}(\underline{\delta\mathcal{U}}); \quad \mathcal{C}^{-1}\,\delta\mathcal{C} = \mathcal{W}(\underline{\delta\mathcal{U}}^*), \tag{34}$$

where the components of the *virtual motion vector* are defined as

$$\underline{\delta\mathcal{U}} = \left| \begin{matrix} \underline{\delta u} \\ \underline{\delta\psi} \end{matrix} \right| \quad \text{and} \quad \underline{\delta\mathcal{U}}^* = \left| \begin{matrix} \underline{\delta u}^* \\ \underline{\delta\psi}^* \end{matrix} \right|, \tag{35}$$

in the reference and material frames, respectively. The components of the virtual rotation and displacement vectors, both measured in the material frame, are $\widetilde{\delta\psi}^* = R^T \delta R$ and $\underline{\delta u}^* = R^T \underline{\delta u}$, respectively. It is now clear that

$$\underline{\delta\mathcal{U}} = \mathcal{C}\,\underline{\delta\mathcal{U}}^*; \quad \underline{\delta\mathcal{U}}^* = \mathcal{C}^{-1}\,\underline{\delta\mathcal{U}}. \tag{36}$$

Similar developments lead to $\mathcal{C}\,\delta\mathcal{C}^{-1} = -\mathcal{W}(\underline{\delta\mathcal{U}})$ and $\delta\mathcal{C}^{-1}\,\mathcal{C} = -\mathcal{W}(\underline{\delta\mathcal{U}}^*)$.

Taking a variation of Equation (32) and a time derivative of Equation (34) leads to

$$\mathcal{W}(\underline{\delta\mathcal{V}}) = \delta\dot{\mathcal{C}}\,\mathcal{C}^{-1} + \dot{\mathcal{C}}\,\delta\mathcal{C}^{-1} \quad \text{and} \quad \mathcal{W}(\underline{\delta\dot{\mathcal{U}}}) = \delta\dot{\mathcal{C}}\,\mathcal{C}^{-1} + \delta\mathcal{C}\,\dot{\mathcal{C}}^{-1}, \tag{37}$$

respectively. Subtracting these two equations then yields

$$\mathcal{W}(\underline{\delta\mathcal{V}} - \underline{\delta\dot{\mathcal{U}}}) = -\mathcal{W}(\mathcal{V})\,\mathcal{W}(\underline{\delta\mathcal{U}}) + \mathcal{W}(\underline{\delta\mathcal{U}})\,\mathcal{W}(\mathcal{V}). \tag{38}$$

Expanding this expression then leads to the following important result

$$\delta\underline{\mathcal{V}} = \underline{\delta\dot{\mathcal{U}}} - \mathcal{W}(\mathcal{V})\,\underline{\delta\mathcal{U}}, \tag{39}$$

that relates virtual changes in the velocity vector to the virtual motion vector and its time derivative. The following results are obtained in a similar manner: $\delta\underline{\mathcal{V}}^* = \underline{\delta\dot{\mathcal{U}}}^* + \mathcal{W}(\mathcal{V}^*)\,\underline{\delta\mathcal{U}}^*$, $\delta\underline{\mathcal{V}} = \mathcal{C}\,\underline{\delta\dot{\mathcal{U}}}^*$, and $\delta\underline{\mathcal{V}}^* = \mathcal{C}^{-1}\,\underline{\delta\dot{\mathcal{U}}}$.

4. Euler Motion Parameters

The definition of the motion tensor, Equation (6), requires six parameters, three parameters to define the displacement vector $\underline{u}$ and three parameters to define the finite rotation tensor R. The goal of this section is to develop a parameterization of finite motion, i.e. to derive a set of parameters that are convenient to manipulate finite motion operations. Euler parameters [1, 19] and quaternion algebra [20, 21] have been shown to provide an effective tool to manipulate finite rotation operation in an algebraic manner. The vector part of Euler parameters is oriented along the eigenvector of the rotation tensor corresponding to a unit eigenvalue. To generalize Euler parameters to the problem of finite motion, a set of parameters are derived that are parallel to the eigenvector of the motion tensor corresponding to a unit eigenvalue. To ease the algebra, the motion tensor, Equation (6), is expanded to an 8×8 operator using the notations defined for quaternion algebra, see Appendix A,

$$\mathbb{C} = \begin{bmatrix} D(\hat{e}) & S(\hat{u})\, D(\hat{e}) \\ 0 & D(\hat{e}) \end{bmatrix}, \tag{40}$$

where $\hat{e}$ is the unit quaternion representing the finite rotation tensor R, and $\hat{u}^T = \lfloor 0, \underline{u}^T \rfloor$ a non-unit quaternion with a vanishing scalar part. An eigenvector of this expanded motion tensor associated with the unit eigenvalue is

$$\mathbb{N} = \begin{vmatrix} \frac{1}{2} B(\hat{e})\, \hat{u} \\ \hat{e} \end{vmatrix} = \begin{vmatrix} \hat{q} \\ \hat{e} \end{vmatrix}. \tag{41}$$

As discussed in Section 2.4, several expressions for the eigenvector can be found, due to the multiplicity of two of the unit eigenvalue. However, the above expression is convenient because a one to one correspondence exists between quaternions $\hat{u}$ and $\hat{q}$; indeed,

$$\hat{q} = \frac{1}{2}\, B(\hat{e})\, \hat{u} \iff \hat{u} = 2B^T(\hat{e})\, \hat{q}. \tag{42}$$

Since $B(\hat{e})$ is an orthogonal operator, this mapping presents no singularities. The scalar part of $\hat{q}$ is $q_0 = -1/2\, \underline{e}^T \underline{u}$. On the other hand, the scalar part of $\hat{u}$ is $u_0 = 0 = 2\, \hat{e}^T \hat{q}$. This implies that quaternions $\hat{e}$ and $\hat{q}$ are orthogonal to each other, $\hat{e}^T \hat{q} = 0$.

The *Euler motion parameters* are proposed as a convenient parameterization of the motion tensor. They form a *bi-quaternion*

$$\breve{g} = \begin{vmatrix} \hat{q} \\ \hat{e} \end{vmatrix}. \tag{43}$$

It is important to note the redundancy in this representation that required eight parameters instead of the six forming the minimum set. The eight Euler motion parameters are subjected to two constraints: the normality condition for quaternion $\hat{e}$ and the orthogonality of quaternions $\hat{e}$ and $\hat{q}$.

4.1. The Motion Tensor

The motion tensor can be evaluated in terms of the Euler motion parameter as

$$\mathbb{C}(\breve{g}) = \begin{bmatrix} D(\hat{e}) & A(\hat{q})\, B^T(\hat{e}) + A(\hat{e})\, B^T(\hat{q}) \\ 0 & D(\hat{e}) \end{bmatrix}, \tag{44}$$

where identities (A7) and (A5) were used together with Equation (42). Introducing operator $\mathbb{A}$ and $\bar{\mathbb{B}}$ as defined in Equations (B2) and (B3), leads to

$$\mathbb{C}(\breve{g}) = \mathbb{A}(\breve{g})\, \bar{\mathbb{B}}(\breve{g}) = \bar{\mathbb{B}}(\breve{g})\, \mathbb{A}(\breve{g}), \tag{45}$$

a bilinear operation in terms of the Euler motion parameters. Note the parallel between this expression and that for the rotation tensor, Equation (A9). The inverse of the motion tensor is

$$\mathbb{C}^{-1}(\breve{g}) = \begin{bmatrix} D^T(\hat{e}) & B(\hat{q})\, A^T(\hat{e}) + B(\hat{e})\, A^T(\hat{q}) \\ 0 & D^T(\hat{e}) \end{bmatrix}, \tag{46}$$

and hence,

$$\mathbb{C}^{-1}(\breve{g}) = \bar{\mathbb{A}}(\breve{g})\,\mathbb{B}(\breve{g}) = \mathbb{B}(\breve{g})\,\bar{\mathbb{A}}(\breve{g}). \tag{47}$$

4.2. The Velocity Vector

The components of the velocity vector measured in the fixed frame are obtained from their definition, Equation (32). This definition is expressed by means of 8×8 operators to ease the algebraic manipulations

$$\dot{\mathbb{C}}(\breve{g})\,\mathbb{C}^{-1}(\breve{g}) = \left[\mathbb{A}(\breve{g})\,\bar{\mathbb{B}}(\breve{g})\right]^{\cdot}\,\mathbb{B}(\breve{g})\,\bar{\mathbb{A}}(\breve{g}) = \mathbb{A}(\dot{\breve{g}})\,\bar{\mathbb{A}}(\breve{g}) + \bar{\mathbb{B}}(\dot{\breve{g}})\,\mathbb{B}(\breve{g}), \tag{48}$$

where identities (B4) and (B7) were used. Since $\bar{\mathbb{B}}\,\mathbb{B} = \mathbb{I}$, $\dot{\bar{\mathbb{B}}}\,\mathbb{B} = -\bar{\mathbb{B}}\,\dot{\mathbb{B}}$, and the previous expression becomes

$$\dot{\mathbb{C}}(\breve{g})\,\mathbb{C}^{-1}(\breve{g}) = \mathbb{A}(\dot{\breve{g}})\,\bar{\mathbb{A}}(\breve{g}) - \bar{\mathbb{B}}(\breve{g})\,\mathbb{B}(\dot{\breve{g}}) = \mathbb{W}(\breve{v}), \tag{49}$$

where $\breve{v}^T = \lfloor \hat{v}^T, \hat{\omega}^T \rfloor$ is the velocity bi-quaternion in the fixed frame. Identities (A5) and (A6) are used to obtain $\hat{v} = 2B^T(\hat{q})\,\dot{\hat{e}} + 2B^T(\hat{e})\,\dot{\hat{q}}$ and $\hat{\omega} = 2B^T(\hat{e})\,\dot{\hat{e}}$. These results are written in a compact manner as

$$\breve{v} = 2\bar{\mathbb{B}}(\breve{g})\,\dot{\breve{g}}. \tag{50}$$

The velocity vector becomes a bilinear expression in terms of the Euler motion parameters and their derivatives. The vector parts of quaternions $\hat{v}$ and $\hat{\omega}$ are the velocity and angular velocity vectors, respectively. The scalar part of the velocity quaternion is $v_0 = 2[\hat{e}^T\dot{\hat{q}} + \hat{q}^T\dot{\hat{e}}] = 0$, since $\hat{q}$ and $\hat{e}$ are orthogonal quaternions. The scalar part of the angular velocity quaternion $\omega_0 = 2\hat{e}^T\dot{\hat{e}} = 0$, since $\hat{e}$ is a unit quaternion. The components of the velocity vector measured in the material frame are obtained in a similar manner

$$\mathbb{C}^{-1}(\breve{g})\,\dot{\mathbb{C}}(\breve{g}) = \mathbb{W}(\breve{v}^*), \tag{51}$$

where $\breve{v}^{*T} = \lfloor \hat{v}^{*T}, \hat{\omega}^{*T} \rfloor$ is the velocity bi-quaternion in the material frame. These results are written in a compact manner as

$$\breve{v}^* = 2\bar{\mathbb{A}}(\breve{g})\,\dot{\breve{g}}, \tag{52}$$

4.3. Composition of Finite Motions

Let $\breve{g}^T = \lfloor \hat{q}^T, \hat{e}^T \rfloor$, $\breve{g}_1^T = \lfloor \hat{q}_1^T, \hat{e}_1^T \rfloor$, and $\breve{g}_2^T = \lfloor \hat{q}_2^T, \hat{e}_2^T \rfloor$ be the bi-quaternions of three motion tensors such that

$$\mathbb{C}(\breve{g}) = \mathbb{C}(\breve{g}_1)\,\mathbb{C}(\breve{g}_2). \tag{53}$$

The problem at hand is to determine bi-quaternion $\breve{g}$ as a function of the other two. With the help of Equation (45), this expands to

$$\mathbb{A}(\breve{g})\,\bar{\mathbb{B}}^T(\breve{g}) = \mathbb{A}(\breve{g}_1)\,\bar{\mathbb{B}}(\breve{g}_1)\,\mathbb{A}(\breve{g}_2)\,\bar{\mathbb{B}}(\breve{g}_2) = \mathbb{A}(\breve{g}_1)\,\mathbb{A}(\breve{g}_2)\,\bar{\mathbb{B}}(\breve{g}_1)\,\bar{\mathbb{B}}(\breve{g}_2), \tag{54}$$

where identity (B4) was used. Expanding the products and using identities (A5) and (A6) then implies $\hat{q} = A(\hat{e}_1)\,\hat{e}_2 + A(\hat{q}_1)\,\hat{e}_2$ and $\hat{e} = A(\hat{e}_1)\,\hat{e}_2$. These results are written in a compact manner as

$$\breve{g} = \mathbb{A}(\breve{g}_1)\,\breve{g}_2 = \mathbb{B}(\breve{g}_2)\,\breve{g}_1. \tag{55}$$

Here again, this operation is bilinear in terms of the Euler motion parameters of the two motions. It is readily shown that $\breve{g}$ is also a bi-quaternion that represents a frame change. Indeed, $\hat{e}^T\hat{e} = \hat{e}_2^T A^T(\hat{e}_1)\, A(\hat{e}_1)\, \hat{e}_2 = \hat{e}_2^T \hat{e}_2 = 1$, since $\hat{e}_1$ and $\hat{e}_2$ both are unit quaternions. Furthermore, $\hat{e}^T\hat{q} = \hat{e}_2^T A^T(\hat{e}_1)\, A(\hat{e}_1)\, \hat{q}_2 + \hat{e}_2^T A^T(\hat{e}_1)\, A(\hat{q}_1)\, \hat{e}_2 = \hat{e}_2^T \hat{q}_2 + \hat{e}_1^T B^T(\hat{e}_2)\, B(\hat{e}_2)\, \hat{q}_1 = \hat{e}_2^T \hat{q}_2 + \hat{e}_1^T \hat{q}_1 = 0$, since $(\hat{e}_1, \hat{q}_1)$ and $(\hat{e}_2, \hat{q}_2)$ are orthogonal quaternions.

4.4. Determination of the Euler Motion Parameters

Finally, an expression for the Euler motion parameters in terms of the components of the motion tensor is sought. Unfortunately, this inverse relationship cannot be expressed in a simple manner. In view of Equation (45), the motion tensor written in the following form

$$\mathcal{C}(\breve{g}) = \begin{bmatrix} R(\hat{e}) & Z(\hat{q}, \hat{e}) \\ 0 & R(\hat{e}) \end{bmatrix}, \tag{56}$$

where $R(\hat{e})$ is given by [7, equation (51)] and

$$Z(\hat{q}, \hat{e}) = 2\left[e_0\widetilde{q} + q_0\widetilde{e} + \widetilde{e}\,\widetilde{q} + \widetilde{q}\,\widetilde{e}\right]. \tag{57}$$

The determination of the bi-quaternion $\breve{g} = \lfloor \hat{q}, \hat{e} \rfloor$ proceeds in two steps. At first, quaternion $\hat{e}$ is determined from the the finite rotation tensor $R(\hat{e})$ by following the procedure described in appendix A of [7].

The second step is to determine quaternion $\hat{q}$ from operator $Z(\hat{q}, \hat{e})$. Consider the following symmetric matrix constructed from the components of $Z(\hat{q}, \hat{e})$

$$T = \begin{bmatrix} \mathrm{tr}(Z) & Z_{32} - Z_{23} & Z_{13} - Z_{31} & Z_{21} - Z_{12} \\ Z_{32} - Z_{21} & & & \\ Z_{13} - Z_{31} & & Z + Z^T - \mathrm{tr}(Z)\, I & \\ Z_{21} - Z_{12} & & & \end{bmatrix} = 4\,(\hat{e}\,\hat{q}^T + \hat{q}\,\hat{e}^T). \tag{58}$$

Quaternion $\hat{q}$ can readily be computed from any column of this matrix. However, this determination will involve a division by components of quaternion $\hat{e}$; hence, inaccurate results will be obtained if dividing by small, or zero values. The most accurate results will be obtained by selecting index m such that $|e_m| > |e_i|$, $i \neq m$. The components of $\hat{q}$ are then

$$q_m = \frac{1}{e_m}\left[\frac{T_{mm}}{8}\right]; \quad q_i = \frac{1}{e_m}\left[\frac{T_{mi}}{4} - e_i q_m\right], \quad i \neq m. \tag{59}$$

Of course, the integrity of the data should be checked by verifying that $\hat{q}$ and $\hat{e}$ are orthogonal quaternions.

5. The Vector Parameterization of Motion

In the previous section, the Euler motion parameters were shown to provide an elegant, purely algebraic representation of finite motion. In fact, when using bi-quaternions, all finite motion operations become bi-linear expressions of various bi-quaternions. However, these advantages come at a tremendous cost: eight parameters must be used instead of six, i.e. the Euler motion parameters do not form a minimum set. Furthermore, the normality and orthogonality conditions inherent to the representation must be enforced as external constraints.

The *vector parameterization of motion* consists of a minimal set of parameters defining of the components of a *motion parameter array*

$$\underline{\mathcal{G}} = \begin{vmatrix} \underline{q} \\ \underline{p} \end{vmatrix} = \begin{vmatrix} D\,\underline{u} \\ \underline{p} \end{vmatrix}. \tag{60}$$

Each of the vectors $\underline{q}$ and $\underline{p}$ follow the transformation rules for first order tensors when subjected to a change of basis. All parameterizations in the class feature *motion parameter arrays that are eigenvectors of the motion tensor corresponding to the eigenvalue* $+1$. However, as discussed in Section 2.4, this eigenvector is not uniquely defined, due to the multiplicity of two of the eigenvalue $\lambda = +1$. Note that the bottom three components of the eigenvector are uniquely defined (within a multiplicative constant), whereas multiple solutions arise for the top three components. In the following sections, the motion parameter arrays will be based on a value of $\alpha = \nu p' - 1$, which corresponds to $D = H^{-1}$, or

$$\underline{\mathcal{G}}_1 = \begin{vmatrix} H^{-1}\underline{u} \\ \underline{p} \end{vmatrix}. \tag{61}$$

However, it should be noted that similar developments could be based on any value of the arbitrary coefficient α.

5.1. The Motion Tensor

The explicit expression for the motion tensor in terms of motion parameters is obtained from Equation (6)

$$\mathcal{C}(\underline{\mathcal{G}}) = \begin{bmatrix} R(\underline{p}) & \widetilde{H(\underline{p})\,\underline{q}}\,R(\underline{p}) \\ 0 & R(\underline{p}) \end{bmatrix}. \tag{62}$$

where $R(\underline{p})$ and $H(\underline{p})$ are given by equations (14) and (22) in [7], respectively.

5.2. The Velocity Vector

The velocity vector is obtained from a time derivative of the motion tensor, as indicated in Equation (26), and writes

$$\underline{\mathcal{V}} = \begin{vmatrix} \underline{v} \\ \underline{\omega} \end{vmatrix} = \begin{vmatrix} \underline{\dot{u}} + \tilde{u}\,\underline{\omega} \\ \underline{\omega} \end{vmatrix} = \begin{vmatrix} \dot{H}(\underline{p})\,\underline{q} + H(\underline{p})\,\underline{\dot{q}} + \widetilde{H(\underline{p})\,\underline{q}}\,H(\underline{p})\,\underline{\dot{p}} \\ H(\underline{p})\,\underline{\dot{p}} \end{vmatrix}. \tag{63}$$

The operator $L(\underline{q}, \underline{p})$ is implicitly defined by the following relationship: $\dot{H}(\underline{p})\,\underline{q} = L(\underline{q}, \underline{p})\,\underline{\dot{p}}$. The velocity vector now becomes

$$\underline{\mathcal{V}} = \begin{bmatrix} H(\underline{p}) & L(\underline{q}, \underline{p}) + \widetilde{H(\underline{p})\,\underline{q}}\,H(\underline{p}) \\ 0 & H(\underline{p}) \end{bmatrix} \begin{vmatrix} \underline{\dot{p}} \\ \underline{\dot{q}} \end{vmatrix} \tag{64}$$

Operator $\mathcal{H}$ is defined as

$$\mathcal{H} = \begin{bmatrix} H(\underline{p}) & L(\underline{q}, \underline{p}) + \widetilde{H(\underline{p})\,\underline{q}}\,H(\underline{p}) \\ 0 & H(\underline{p}) \end{bmatrix}, \tag{65}$$

and the velocity vector becomes

$$\underline{\mathcal{V}} = \mathcal{H}\,\dot{\underline{\mathcal{q}}}. \tag{66}$$

Similar developments lead to

$$\underline{\mathcal{V}}^* = \mathcal{H}^*\,\dot{\underline{\mathcal{q}}}, \tag{67}$$

where

$$\mathcal{H}^* = \begin{bmatrix} H^T(\underline{p}) & R^T L(\underline{q},\underline{p}) \\ 0 & H^T(\underline{p}) \end{bmatrix}. \tag{68}$$

The inverse of these operators are readily found as

$$\mathcal{H}^{-1} = \begin{bmatrix} H^{-1}(\underline{p}) & -H^{-1}\left[L(\underline{q},\underline{p})\,H^{-1} + \widetilde{H(\underline{p})\,\underline{q}}\,H(\underline{p})\right] \\ 0 & H^{-1}(\underline{p}) \end{bmatrix}, \tag{69}$$

and

$$\mathcal{H}^{*-1} = \begin{bmatrix} H^{-T}(\underline{p}) & -H^{-1}L(\underline{q},\underline{p})H^{-T} \\ 0 & H^{-T}(\underline{p}) \end{bmatrix}. \tag{70}$$

The operators enjoy the following remarkable properties

$$\mathcal{C} = \mathcal{H}\,\mathcal{H}^{*-1}; \quad \mathcal{C}^{-1} = \mathcal{H}^*\,\mathcal{H}^{-1}. \tag{71}$$

that echo the corresponding properties for the rotation tensor, see [7, equation (28)].

5.3. Determination of the Motion Parameters

The motion tensor, Equation (62) can be written as

$$\mathcal{C}(\underline{\mathcal{q}}) = \begin{bmatrix} R(\underline{p}) & Z(\underline{p},\underline{q}) \\ 0 & R(\underline{p}) \end{bmatrix}. \tag{72}$$

To determine the components of the vector parameterization from this motion tensor, the vector parameterization of $R(\underline{p})$ is extracted first by following the procedure described in [7]. Then array $\underline{q}$ is extracted from Z

$$\underline{q} = H^{-1}(\underline{p})\,\mathrm{axial}\left[Z(\underline{p},\underline{q})\,R^T(\underline{p})\right]. \tag{73}$$

5.4. Composition of Finite Motions

Let $\underline{\mathcal{q}}$, $\underline{\mathcal{q}}_1^T = \lfloor \underline{q}_1^T, \underline{p}_1^T \rfloor$, and $\underline{\mathcal{q}}_2^T = \lfloor \underline{q}_2^T, \underline{p}_2^T \rfloor$ correspond to motion tensors $\mathcal{C}(\underline{\mathcal{q}})$, $\mathcal{C}(\underline{\mathcal{q}}_1)$, and $\mathcal{C}(\underline{\mathcal{q}}_2)$, respectively. If $\mathcal{C}(\underline{\mathcal{q}}) = \mathcal{C}(\underline{\mathcal{q}}_1)\,\mathcal{C}(\underline{\mathcal{q}}_2)$, the problem is to relate is $\underline{\mathcal{q}}$ to $\underline{\mathcal{q}}_1$ and $\underline{\mathcal{q}}_2$. The first step of the process is to note that $R = R_1 R_2$, and hence, equations (18) and (19) in [7] yield $\underline{p}$ as a function of $\underline{p}_1$ and $\underline{p}_2$. Next, the relationship between $\underline{q}$ and $\underline{q}_1$, $\underline{q}_2$ is obtained as

$$\underline{q} = H^{-1}(\underline{p})\,H(\underline{p}_1)\,\underline{q}_1 + H^{-T}(\underline{p})\,H^T(\underline{p}_2)\,\underline{q}_2. \tag{74}$$

6. Special Choices of the Generating Function

The vector parameterization of motion presented in the previous section, Equation (60), consists of a set of displacement related parameters, $\underline{q} = D\,\underline{u}$, and of the vector parameterization of rotations, $\underline{p}$, forming an eigenvector of the motion tensor corresponding to its positive unit eigenvalue. This leads to families of parameterizations that depend on two choices: the choice of the generating function determining $\underline{p}$, and on the choice of the arbitrary coefficient α.

As discussed in [7], the generating function, $p(\phi)$, can be selected to simplify some of the operators involved in manipulating rotations. But more importantly, judicious choices of this function can eliminate singularities that occur in the various rotation operators. The occurrence of singularities is also a major concern when dealing with the vector parameterization of motion: a one to one, singularity free relationship must exist between the motion parameters $\underline{q} = D\,\underline{u}$ and the physical displacements $\underline{u}$. In turn, this implies that $\det(D) = (1+\alpha)/\nu^3 \neq 0$ or ∞. While this condition does not determine α, it limits the possible choices.

A simple choice is $\alpha = 0$. This implies $D = 1/\nu\ G^T$ and $\det(D) = 1/\nu^3$. A second choice is $\alpha = \nu p' - 1$. This implies $D = H^{-1}$ and $\det(D) = \det(H^{-1}) = p'/\nu^2$. The choice $\alpha = \nu/\varepsilon$ leads to a particularly simple form $D = 1/\varepsilon - \tilde{p}/2$, where the quadratic term in $\underline{p}$ vanishes; in this case, $\det(D) = 1/(\varepsilon\nu^2)$. The next possible choice is $\alpha = \nu/R_1 - 1$ leading to $D = (I + R^T)/2R_1$ and $\det(D) = 1/(\nu^2 R_1)$. A final choice, $\alpha = \nu^3 - 1$, gives $D = \nu^2 - \tilde{p}/2 + (\nu^2 - 1/\varepsilon)\,\tilde{p}\tilde{p}/p^2$ and $\det(D) = 1$.

Singularity-free rotation parameterizations [7] require $\nu \neq 0$ or ∞, and $\det(H^{-1}) \neq 0$ or ∞. Consequently, the first two choices of α leads to singularity-free parameterizations of motion when singularity-free rotation parameterizations are used. The next two choices lead to additional requirements, $\varepsilon \neq 0$ or ∞, and $R_1 \neq 0$ or ∞ to avoid singularities; this, in turn, would limit the validity range of these parameterizations to $|\phi| < \pi$. The last choice clearly is singularity-free, but involves an operator D that has no simple relationship to other rotation related operators. Other choices of α might lead to singularity-free parameterizations of motion but are likely to involve an operator D that is unrelated to other rotation related operators.

Clearly, the first two choices of α appear to be most desirable. Furthermore, if the Cayley–Gibbs–Rodrigues parameters are selected for the rotation parameterization, $\alpha = \nu p' - 1 = \nu/R_1 - 1$ and if the Wiener–Milenkovic parameters are used, $\alpha = \nu p' - 1 = 0$. In other words, $\alpha = \nu/R_1 - 1$ and $\alpha = 0$ are special cases of $\alpha = \nu p' - 1$ for specific choices of the rotation parameterization. In view of this discussion, the optimum choice seems to be $\alpha = \nu p' - 1$, leading to the motion parameters defined by Equation (61).

7. Specific Parameterizations of Motion

Once the form of the eigenvector of the motion tensor has been selected as Equation (61), parameterizations of motion then only depend on the choice of the generating function, $p(\phi)$. Many alternatives choices of this function and their respective advantages and drawbacks were discussed in [5, 7], where two parameterizations were shown to present unique properties: the Cayley–Gibbs–Rodrigues and the Wiener–Milenkovic parameterization that will be presented in detail in the following sections. One of the simplest choices of the generating function is $p(\phi) = \phi$ that leads to the exponential map of motion [18]. This parameterization involves the evaluation of numerous trigonometric functions and will not be discussed here, although all the relevant formulæ can be obtained by introducing the generating function $p(\phi) = \phi$ into the expression given in the previous section.

7.1. The Cayley–Gibbs–Rodrigues Motion Parameters

The Cayley–Gibbs–Rodrigues motion parameters are defined as

$$\underline{\mathcal{g}} = \begin{bmatrix} H^{-1}\underline{u} \\ \underline{r} \end{bmatrix} = \begin{bmatrix} \frac{1}{r_0}\frac{I + R^T}{2}\underline{u} \\ \underline{r} \end{bmatrix}, \tag{75}$$

where $\underline{r} = 2\tan\phi/2\,\bar{n}$ are the Cayley–Gibbs–Rodrigues parameters of the finite rotation tensor, see [7], and $r_0 = 1/(1 + \underline{r}^T\underline{r}/4)$. This motion parameterization is valid for rotations of magnitude $|\phi| < \pi$, since the Cayley–Gibbs–Rodrigues parameters suffer this limitation.

7.1.1. *The Motion Tensor*

The motion tensor expressed in terms of Cayley–Gibbs–Rodrigues motion parameters is found from Equation (6)

$$\mathcal{C} = \begin{bmatrix} R & \widetilde{H\underline{q}}\,R \\ 0 & R \end{bmatrix} = \begin{bmatrix} R & \frac{R+I}{2}\tilde{q}\frac{R+I}{2} \\ 0 & R \end{bmatrix}. \tag{76}$$

The inverse of the motion tensor is readily obtained as

$$\mathcal{C}^{-1} = \begin{bmatrix} R^T & \frac{R^T+I}{2}\tilde{q}^T\frac{R^T+I}{2} \\ 0 & R^T \end{bmatrix}. \tag{77}$$

Cayley's decomposition applies to this parameterization, i.e.

$$\mathcal{C} = \left[\mathcal{I} + \frac{1}{2}\mathcal{W}(\underline{\mathcal{g}})\right]\left[\mathcal{I} - \frac{1}{2}\mathcal{W}(\underline{\mathcal{g}})\right]^{-1} = \left[\mathcal{I} - \frac{1}{2}\mathcal{W}(\underline{\mathcal{g}})\right]^{-1}\left[\mathcal{I} + \frac{1}{2}\mathcal{W}(\underline{\mathcal{g}})\right], \tag{78}$$

which generalizes the well-known decomposition of the rotation tensor [7, equation (95)]. This important decomposition of the motion tensor could be taken as the basis for the determination of this specific parameterization [18].

7.1.2. *The Velocity Vector*

Operators $\mathcal{H}$, $\mathcal{H}^*$ and their inverses are obtained from Equations (65), (68), (69), and (70), respectively. First, operator $L(\underline{q}, \underline{p})$ is found to be

$$L(\underline{q}, \underline{r}) = -\frac{r_0}{2}\left[\tilde{q} + H\,\underline{q}\,\underline{r}^T\right]. \tag{79}$$

It then follows that

$$\mathcal{H} = \begin{bmatrix} H & \frac{r_0}{2}[\tilde{q} - (\underline{r}^T\underline{q})H] \\ 0 & H \end{bmatrix}, \tag{80}$$

$$\mathcal{H}^* = \begin{bmatrix} H^T & \frac{r_0}{2}[\tilde{q}^T - (\underline{r}^T\underline{q})H^T] \\ 0 & H^T \end{bmatrix}, \tag{81}$$

$$\mathcal{H}^{-1} = \begin{bmatrix} H^{-1} & \frac{r_0}{2}[H^{-1}\tilde{q}^T H^{-1} + (\underline{r}^T \underline{q}) H^{-1}] \\ 0 & H^{-1} \end{bmatrix}, \tag{82}$$

$$\mathcal{H}^{*-1} = \begin{bmatrix} H^{-T} & \frac{r_0}{2}[H^{-T}\tilde{q} H^{-T} + (\underline{r}^T \underline{q}) H^{-T}] \\ 0 & H^{-T} \end{bmatrix}. \tag{83}$$

7.1.3. *Composition of Motions*

The composition of motion follows the general procedure described in section 5.4: the parameters $\underline{r}$ of the relative rotation are first determined from the formulæ for the composition of rotation [7, equations (18) and (19)]; next, parameters $\underline{q}$ are evaluated with the help of Equation (74).

7.2. THE WIENER–MILENKOVIC MOTION PARAMETERS

The Wiener–Milenkovic motion parameters are defined as

$$\underline{\mathcal{g}} = \begin{bmatrix} H^{-1}\underline{u} \\ \underline{c} \end{bmatrix} = \begin{bmatrix} \frac{1}{\nu} G^T \underline{u} \\ \underline{c} \end{bmatrix}, \tag{84}$$

where $\underline{c} = 4 \tan \phi/4\, \bar{n}$ are the Wiener–Milenkovic [22] parameters of the rotation tensor. This parameterization of rotation is singularity free for rotations of arbitrary magnitude when using the rescaling technique [7]. Consequently, the Wiener–Milenkovic motion parameterization is singularity free for displacements and rotations of arbitrary magnitude provided that the rescaling operation is applied to the rotation parameterization.

7.2.1. *The Motion Tensor*

The motion tensor expressed in terms of Wiener–Milenkovic motion parameters is found from Equation (6)

$$\mathcal{C} = \begin{bmatrix} R & \widetilde{H\underline{q}}\, R \\ 0 & R \end{bmatrix} = \begin{bmatrix} R & G\, \widetilde{\nu q}\, G \\ 0 & R \end{bmatrix} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} I & \widetilde{\nu q} \\ 0 & I \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}. \tag{85}$$

The factorization of the motion tensor affords the following geometric interpretation: the motion is decomposed into the half-angle rotation characterized by the rotation tensor G, followed by a translation of magnitude $\nu\underline{q}$, and finally a half-angle rotation. The inverse of the motion tensor is readily obtained as

$$\mathcal{C}^{-1} = \begin{bmatrix} R^T & G^T\, \widetilde{\nu q}^T\, G^T \\ 0 & R^T \end{bmatrix}, \tag{86}$$

7.2.2. *The Velocity Vector*

Operators $\mathcal{H}$, $\mathcal{H}^*$ and their inverses are obtained from Equations (65), (68), (69), and (70), respectively. First, operator $L(\underline{q}, \underline{p})$ is found to be

$$L(\underline{q}, \underline{c}) = -\frac{\nu^2}{2} G \left[\widetilde{\left(1 + \frac{\tilde{c}}{4}\right)} \underline{q} + \frac{1}{4}(\underline{c}^T \underline{q}) \right]. \tag{87}$$

It then follows that

$$\mathcal{H} = \begin{bmatrix} H & \frac{\nu}{2} H \left[\widetilde{\left(1 - \frac{\tilde{c}}{4}\right) \underline{q}} - \frac{1}{4}(\underline{c}^T \underline{q}) \right] \\ 0 & H \end{bmatrix}, \tag{88}$$

$$\mathcal{H}^* = \begin{bmatrix} H^T & \frac{\nu}{2} H^T \left[\widetilde{\left(1 + \frac{\tilde{c}}{4}\right) \underline{q}}^T - \frac{\underline{c}^T \underline{q}}{4} \right] \\ 0 & H^T \end{bmatrix}, \tag{89}$$

$$\mathcal{H}^{-1} = \begin{bmatrix} H^{-1} & \frac{\nu}{2} \left[\widetilde{\left(1 - \frac{\tilde{c}}{4}\right) \underline{q}}^T + \frac{\underline{c}^T \underline{q}}{4} \right] H^{-1} \\ 0 & H^{-1} \end{bmatrix}, \tag{90}$$

$$\mathcal{H}^{*-1} = \begin{bmatrix} H^{-T} & \frac{\nu}{2} \left[\widetilde{\left(1 + \frac{\tilde{c}}{4}\right) \underline{q}} + \frac{\underline{c}^T \underline{q}}{4} \right] H^{-T} \\ 0 & H^{-T} \end{bmatrix}. \tag{91}$$

7.2.3. *Composition of Motions*

The composition of motion follows the general procedure described in Section 5.4: the parameters $\underline{r}$ of the relative rotation are first determined from the formulæ for the composition of rotation [7, equations (18) and (19)]; next, parameters $\underline{q}$ are evaluated with the help of Equation (74).

8. Numerical Example

Consider the rigid body depicted in Figure 1. Its velocity in the material frame is given by Equation (10). The kinetic energy of the rigid body is $K = 1/2\ \underline{\mathcal{V}}^{*T} \mathcal{M}^* \underline{\mathcal{V}}^*$, where the mass matrix in the material frame is

$$\mathcal{M}^* = \mu \begin{bmatrix} I & \tilde{\eta}^{*T} \\ \tilde{\eta}^* & \iota^* \end{bmatrix}. \tag{92}$$

The following notation was used: μ is the total mass of the body; $\underline{\eta}^*$ the components the position vector of the center of mass of the rigid body in the material frame $\mathcal{F}$; and $\mu\,\iota^*$ the components the moment of inertia tensor of the rigid body in the material frame $\mathcal{F}$. Hamilton's principle then states that

$$\int_t \left(\delta K + \underline{\delta \mathcal{U}}^{*T}\ \underline{\mathcal{F}}^*\right) \mathrm{d}t = 0, \tag{93}$$

where $\underline{\mathcal{F}}^*$, given by Equation (15), is the vector of externally applied loads measured in the material frame. If $\mathcal{C}_0$ is the motion tensor that bring frame $\mathcal{F}^I$ to frame $\mathcal{F}^R$, measured in

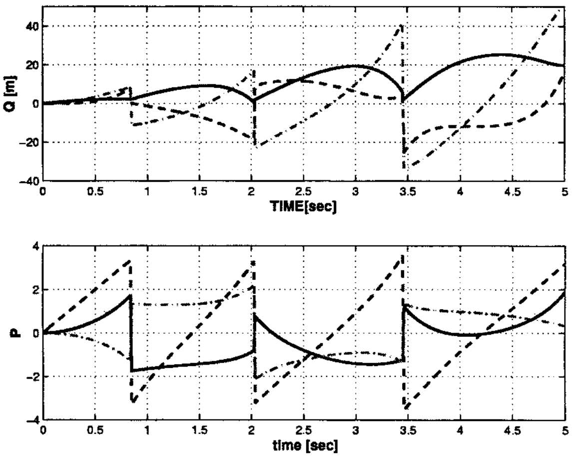

Figure 2. Time history of the Wiener–Milenkovic motion parameters: parameters $\underline{q}$ (top figure), and $\underline{p}$ (bottom figure). Components along $\bar{\imath}_1$, solid line; $\bar{\imath}_2$, dashed line; and $\bar{\imath}_3$, dashed-dotted line.

frame $\mathcal{F}^I$, $\underline{\mathcal{V}} = \mathcal{C}\mathcal{C}_0\,\underline{\mathcal{V}}^*$ then represents the velocity vector in the inertial frame. The virtual velocity vector now becomes

$$\begin{aligned}\delta\underline{\mathcal{V}}^* &= (\mathcal{C}\mathcal{C}_0)^{-1}\left[\delta\underline{\mathcal{V}} + C\delta\mathcal{C}^{-1}\underline{\mathcal{V}}\right]\\ &= (\mathcal{C}\mathcal{C}_0)^{-1}\left[\delta\dot{\underline{\mathcal{U}}} - \mathcal{W}(\underline{\mathcal{V}})\,\delta\underline{\mathcal{U}} - \mathcal{W}(\delta\underline{\mathcal{U}})\,\underline{\mathcal{V}}\right] = (\mathcal{C}\mathcal{C}_0)^{-1}\,\delta\dot{\underline{\mathcal{U}}},\end{aligned} \tag{94}$$

where Equation (39) was used. Hamilton's principle then becomes

$$\int_t \left(\delta\dot{\underline{\mathcal{U}}}^T\,(\mathcal{C}\mathcal{C}_0)^{-T}\,\mathcal{M}^*\,(\mathcal{C}\mathcal{C}_0)^{-1}\,\underline{\mathcal{V}} + \delta\underline{\mathcal{U}}^T\,(\mathcal{C}\mathcal{C}_0)^{-T}\,\underline{\mathcal{F}}^*\right)\mathrm{d}t = 0. \tag{95}$$

The mass matrix of the rigid body in the inertial frame $\mathcal{F}^I$ is $\mathcal{M} = (\mathcal{C}\mathcal{C}_0)^{-T}\,\mathcal{M}^*\,(\mathcal{C}\mathcal{C}_0)^{-1}$. This transformation performs two tasks: a change of reference point from **A** to **O** and a change of orthonormal basis from $\mathcal{B}$ to $\mathcal{B}^I$.

Hamilton's principle now becomes

$$\int_t \left(\delta\dot{\underline{\mathcal{U}}}^T\,\underline{\mathcal{P}} + \delta\underline{\mathcal{U}}^T\,\underline{\mathcal{F}}\right)\mathrm{d}t = 0, \tag{96}$$

where $\underline{\mathcal{P}} = \mathcal{M}\,\underline{\mathcal{V}}$ is the momentum vector and $\underline{\mathcal{F}} = (\mathcal{C}\mathcal{C}_0)^{-T}\,\underline{\mathcal{F}}^*$ the vector of externally applied loads, both measured in the inertial frame. The equations of motion of the rigid body then are

$$\dot{\underline{\mathcal{P}}} = \underline{\mathcal{F}}. \tag{97}$$

This simple example demonstrates the advantages of formulations based on the motion tensor. First, the derivation of the equations of motion is simplified as compared to classical formulations that deal with separate displacement and rotation fields. Second, the equations of motion present a simpler form, as seen in Equation (97). In the absence of externally applied loads, the momentum vector $\underline{\mathcal{P}}$ remains a constant; this is the well known property of conservation of linear and angular momentum of the body.

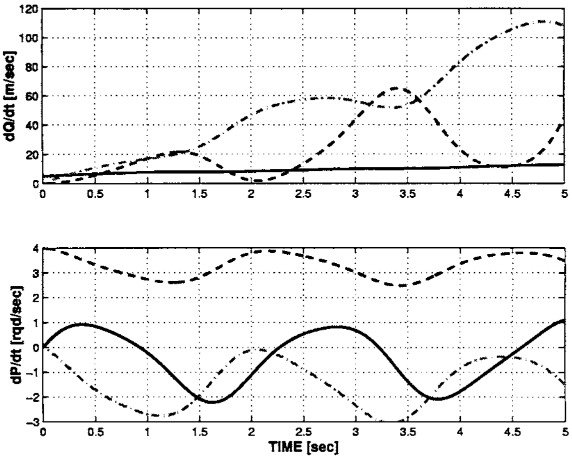

Figure 3. Time history of the time derivatives of the Wiener–Milenkovic motion parameters: parameters $\underline{\dot{q}}$ (top figure), and $\underline{\dot{p}}$ (bottom figure). Components along $\bar{\imath}_1$, solid line; $\bar{\imath}_2$, dashed line; and $\bar{\imath}_3$, dashed-dotted line.

A rigid body with the following properties was simulated: total mass, $\mu = 1.8$ kg, center of mass location, $\underline{\eta}^* = (0.1,\ -0.4,\ 0.3)$ m, moments of inertia, $\mu(\iota^*_{11}, \iota^*_{22}, \iota^*_{33}) = (1.1,\ 0.6,\ 0.9)$ kg.m^2, and $\mu(\iota^*_{12}, \iota^*_{13}, \iota^*_{23}) = (0.012,\ -0.015,\ 0.023)$ kg.m^2. The rigid body had an initial velocity $\underline{v}_0^T = [5,\ 0,\ 0]$ m/sec and initial angular velocity $\underline{\omega}_0^T = [0,\ 4,\ 0]$ rad/sec. The configuration of the rigid body was computed for 5 sec period using a central difference scheme with a constant time step $\Delta t = 0.0125$ sec. The Wiener–Milenkovic motion parameters were used to represent the motion of the body. Figure 2 depicts the evolution of these parameters. The sudden changes in parameter value are associated with rescaling operations that circumvent the singularities associated with the Wiener–Milenkovic parameterization [7]. Figure 3 shows the time derivatives of the parameters which are unaffected by the rescaling operations. The physical displacements and rotations of the body are depicted in Figure 4 and the components of the linear and angular velocity vectors are shown in Figure 5. In these last two figures, the predictions of a classical formulation of the rigid body problem are also given; as expected, excellent correlation is found between the two formulations.

9. Conclusions

This paper developed vector parameterizations of motion that generalizes the well-known parameterizations of rotation. The Plücker coordinates of an arbitrary material line of a rigid body were shown to transform by the action of a motion tensor. The proposed vector parameterization completely describes an arbitrary motion by means of two vectors that constitute an eigenvector of the motion tensor associated with its positive unit eigenvalue.

The parallelism between the proposed vector parameterization of motion and that of rotation is striking. Most techniques and formulæ developed for rotation have their counterparts for motion. In fact, the proposed formulation allows any rotation parameterization to become the basis for a motion parameterization. Several parameterizations were presented for which the treatment of motion is reduced to purely algebraic operations. First, the well known Euler motion parameters that are singularity free and are particularly simple to manipulate but form a redundant set of parameters. Next, the Cayley–Gibbs–Rodrigues motion parameters

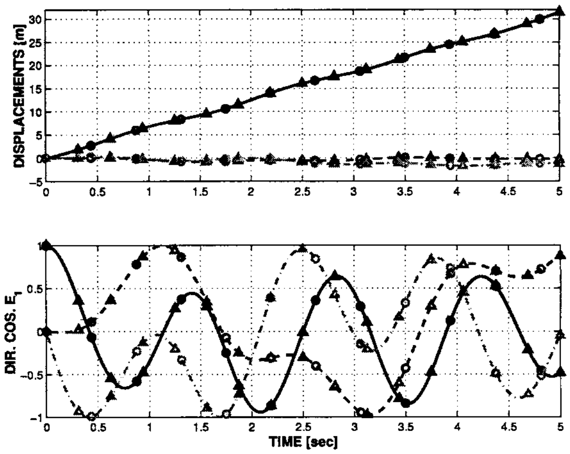

Figure 4. Time history of the physical displacement (top figure) and direction cosines of vector $\underline{E}_1$ (bottom figure). Components along $\bar{\imath}_1$, solid line; $\bar{\imath}_2$, dashed line; and $\bar{\imath}_3$, dashed-dotted line. Solution with Wiener–Milenkovic motion parameters (∘) and classical solution (△).

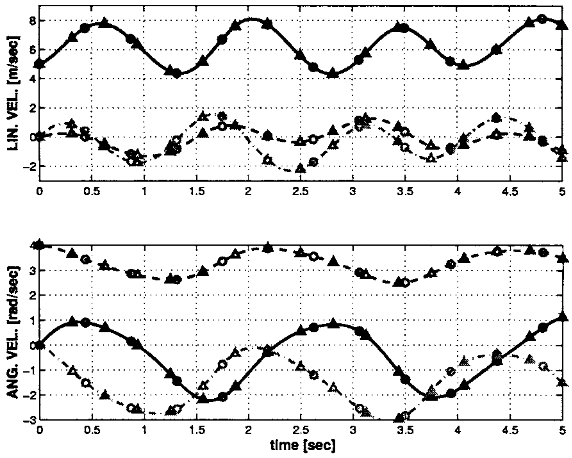

Figure 5. Time history of the linear (top figure) and angular (bottom figure) velocity vectors. Components along $\bar{\imath}_1$, solid line; $\bar{\imath}_2$, dashed line; and $\bar{\imath}_3$, dashed-dotted line. Solution with Wiener–Milenkovic motion parameters (∘) and classical solution (△).

present the desirable Cayley decomposition property, but present a singularity for rotations of $|\phi| = \pi$. Finally, the Wiener–Milenkovic motion parameters offer a singularity free treatment of motion without redundancy. This appear to be the first purely algebraic, singularity free parameterization of finite motion.

With the proposed formalism, any singularity free vector parameterization of rotation gives rise to a singularity free vector parameterization of motion. Furthermore, different families of parameterizations can be obtained by selecting a single parameter α that controls the definition of the displacement related motion parameters. For instance, the purely algebraic, singularity free parameterization of rotation defined as $\underline{p} = 4 \sin \phi/4\, \bar{n}$ proposed in [7] also leads to a purely algebraic, singularity free parameterization of motion.

The vector parameterization of motion is well suited for applications to dynamics. The velocity and virtual motion vectors allow the derivation of the equations of motion of systems whose kinematics are described with the proposed parameterization. Classical formulations of dynamics can be used to this effect. The proposed formalism was presented for rigid bodies but could also be used for the modeling of flexible components whose kinematic description requires both displacement and rotation fields such as beams and shells. The governing equations of motion of for such components will be greatly simplified, as was observed for rigid bodies.

Appendix A. Quaternion Algebra

Quaternion algebra is convenient to express finite rotation operations. A *quaternion* [20] is defined as an array of four number

$$\hat{e} = \begin{bmatrix} e_0 \\ \underline{e} \end{bmatrix}, \tag{A1}$$

e_0 is the *scalar part of the quaternion* and $\underline{e}$ the *vector part of the quaternion*. Note that this four component array is not a vector, as it does not transform like a vector quantity. The norm of quaternion $\hat{p}$ is defined as $|\hat{p}| = \sqrt{\hat{p}^T \hat{p}} = \sqrt{p_0^2 + \underline{p}^T \underline{p}}$. Operations on quaternions are performed by using the following matrices:

$$A(\hat{e}) = \begin{bmatrix} e_0 & -\underline{e}^T \\ \underline{e} & e_0 + \tilde{e} \end{bmatrix}; \quad B(\hat{e}) = \begin{bmatrix} e_0 & -\underline{e}^T \\ \underline{e} & e_0 - \tilde{e} \end{bmatrix}; \tag{A2}$$

Let $\hat{p}$ and $\hat{q}$ be two arbitrary quaternions. The following identities hold:

$$A(\hat{p})\, A^T(\hat{p}) = B(\hat{p})\, B^T(\hat{p}) = C(\hat{p})\, C^T(\hat{p}) = |\hat{p}|^2\, I_4, \tag{A3}$$

where I_4 is the 4×4 identity matrix. The following identity is readily verified and expresses the fact that two of the operators commute:

$$A(\hat{p})\, B^T(\hat{q}) = B^T(\hat{q})\, A(\hat{p}). \tag{A4}$$

These identities then imply the following results $A(\hat{p})\, \hat{q} = B(\hat{q})\, \hat{p}$. Finally, the following results are easily checked:

$$A(\hat{p})\, A(\hat{q}) = A(\hat{r}) \Longleftrightarrow \hat{r} = A(\hat{p})\, \hat{q} = B(\hat{q})\, \hat{p}. \tag{A5}$$

Similarly

$$B(\hat{p})\, B(\hat{q}) = B(\hat{r}) \Longleftrightarrow \hat{r} = B(\hat{p})\, \hat{q} = A(\hat{q})\, \hat{p}. \tag{A6}$$

Finally, the skew-symmetric operator $S(\hat{p})$ is defined as

$$S(\hat{p}) = \begin{bmatrix} 0 & \underline{0}^T \\ \underline{0} & \tilde{p} \end{bmatrix}; \quad S(\hat{p}) = \frac{1}{2}[A(\hat{p}) - B(\hat{p})]. \tag{A7}$$

A quaternion $\hat{e}$ is said to be a *unit quaternion* if its norm is unity, i.e. $|\hat{e}| = 1$. In view of identity (A3), operators $A(\hat{e})$ and $B(\hat{e})$ now become orthogonal matrices. An additional operator is now defined

$$D(\hat{e}) = \begin{bmatrix} 1 & 0 \\ 0 & R(\hat{e}) \end{bmatrix}, \tag{A8}$$

where $R(\hat{e})$ is the rotation operator. It is now readily verified that

$$D(\hat{e}) = A(\hat{e})\, B^T(\hat{e}) = B^T(\hat{e})\, A(\hat{e}). \tag{A9}$$

Appendix B. Bi-Quaternion Algebra

Bi-quaternion algebra is convenient to express finite motion operations. A *bi-quaternion* is defined as an array of two quaternions

$$\check{g} = \begin{vmatrix} \hat{q} \\ \hat{e} \end{vmatrix}, \tag{B1}$$

Operations on bi-quaternions are performed by using a number of matrices defined as follows:

$$\mathbb{A}(\check{g}) = \begin{bmatrix} A(\hat{e}) & A(\hat{q}) \\ 0 & A(\hat{e}) \end{bmatrix}; \quad \bar{\mathbb{A}}(\check{g}) = \begin{bmatrix} A^T(\hat{e}) & A^T(\hat{q}) \\ 0 & A^T(\hat{e}) \end{bmatrix}; \tag{B2}$$

$$\mathbb{B}(\check{g}) = \begin{bmatrix} B(\hat{e}) & B(\hat{q}) \\ 0 & B(\hat{e}) \end{bmatrix}; \quad \bar{\mathbb{B}}(\check{g}) = \begin{bmatrix} B^T(\hat{e}) & B^T(\hat{q}) \\ 0 & B^T(\hat{e}) \end{bmatrix}. \tag{B3}$$

Let $\check{g}^T = \lfloor \hat{q}^T, \hat{e}^T \rfloor$ and $\check{h}^T = \lfloor \hat{r}^T, \hat{f}^T \rfloor$ be two arbitrary bi-quaternions. The following identities hold:

$$\mathbb{A}(\check{g})\, \bar{\mathbb{B}}(\check{h}) = \bar{\mathbb{B}}(\check{h})\, \mathbb{A}(\check{g}). \tag{B4}$$

These identities then imply the following results:

$$\mathbb{A}(\check{g})\, \check{h} = \mathbb{B}(\check{h})\, \check{g}. \tag{B5}$$

The operator $\mathbb{W}$ is defined as

$$\mathbb{W}(\check{g}) = \begin{bmatrix} S(\hat{e}) & S(\hat{q}) \\ 0 & S(\hat{e}) \end{bmatrix}; \quad \mathbb{W}(\check{g}) = \frac{1}{2}[\mathbb{A}(\check{g}) - \mathbb{B}(\check{g})]. \tag{B6}$$

Two quaternions $\hat{q}$ and $\hat{e}$ are said to be orthogonal if $\hat{q}^T \hat{e} = 0$. For such pair of quaternions the following identities can be shown to hold:

$$\bar{\mathbb{A}}(\check{g})\, \mathbb{A}(\check{g}) = \mathbb{A}(\check{g})\, \bar{\mathbb{A}}(\check{g}) = \bar{\mathbb{B}}(\check{g})\, \mathbb{B}(\check{g}) = \mathbb{B}(\check{g})\, \bar{\mathbb{B}}(\check{g}) = |\hat{e}|^2\, \mathbb{I}. \tag{B7}$$

References

1. Stuelpnagel, J., 'On the parameterization of the three-dimensional rotation group', *SIAM Review* **6**(4), 1964, 422–430.
2. Kane, T. R., *Dynamics*, Holt, Rinehart and Winston, New York, 1968.
3. Hughes, P. C., *Spacecraft Attitude Dynamics*, Wiley, New York, 1986.
4. Cardona, A., 'An integrated approach to mechanism analysis', Ph.D. Thesis, Université de Liège, 1989.
5. Shuster, M. D., 'A survey of attitude representations', *Journal of the Astronautical Sciences* **41**, 1993, 439–517.
6. Ibrahimbegović, A., 'On the choice of finite rotation parameters', *Computer Methods in Applied Mechanics and Engineering* **149**, 1997, 49–71.

7. Bauchau, O. A. and Trainelli, L., 'The vectorial parameterization of rotation', *Nonlinear Dynamics* **32**(1), 2003, 71–92.
8. Bottema, O. and Roth, B., *Theoretical Kinematics*, Dover, New York, 1979.
9. Angeles, J., *Fundamentals of Robotic Mechanical Systems. Theory, Methods, and Algorithms*, Springer, New York, 1997.
10. Angeles, J., 'The application of dual algebra to kinematic analysis', in *Computational Methods in Mechanical Systems*, J. Angeles and E. Zakhariev (eds.), Springer, Heidelberg, 1998, pp. 3–31.
11. Yang, A. T. and Freudenstein, F., 'Application of dual-number quaternion algebra to the analysis of spatial mechanims', *ASME Journal of Applied Mechanics* **86**, 1964, 300–308.
12. Agrawal, O. P., 'Hamilton operators and dual-number quaternions in spatial kinematics', *Mechanism and Machine Theory* **22**(6), 1987, 569–575.
13. McCarthy, J. M., *An Introduction to Theoretical Kinematics*, The MIT Press, Cambridge, MA, 1990.
14. Fischer, I. S., *Dual Number Methods in Kinematics, Statics and Dynamics*, CRC Press, Boca Raton, FL, 1999.
15. Pradeep, A. K., Yoder, P. J., and Mukundan, R., 'On the use of dual matrix exponentials in robot kinematics', *International Journal of Robotics Research* **8**(5), 1989, 57–66.
16. Brodsky, V. and Shoham, M., 'The dual inertia operator and its application to robot dynamics', *Journal of Mechanical Design* **116**, 1994, 1089–1095.
17. Chevallier, D. P., 'Lie algebra, modules, dual quaternions and algebraic methods in kinematics', *Mechanism and Machine Theory* **26**(6), 1991, 613–627.
18. Borri, M., Trainelli, L., and Bottasso, C. L., 'On representations and parameterizations of motion', *Multibody Systems Dynamics* **4**, 2000, 129–193.
19. Spring, K. W., 'Euler parameters and the use of quaternion algebra in the manipulation of finite rotations: A review', *Mechanism and Machine Theory* **21**, 1986, 365–373.
20. Hamilton, W. R., *Elements of Quaternions*, Cambridge University Press, Cambridge, 1899.
21. Wehage, R. A., 'Quaternions and Euler parameters. A brief exposition', in *Computer Aided Analysis and Optimization of Mechanical Systems Dynamics*, E. J. Haug (ed.), Springer, Berlin, 1984.
22. Milenkovic, V., 'Coordinates suitable for angular motion synthesis in robots', in *Proceedings of the Robot VI Conference*, Detroit, MI, March 2–4, 1982, Paper MS82-217.