Three-Dimensional Beam Theory for Flexible Multibody Dynamics*

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Abstract

In multibody systems, it is common practice to approximate flexible components as beams or shells. More often than not, classical beam theories, such as Euler-Bernoulli beam theory, form the basis of the analytical development for beam dynamics. The advantage of this approach is that it leads to simple kinematic representations of the problem: the beam's section is assumed to remain plane and its displacement field is fully defined by three displacement and three rotation components. While such approach is capable of capturing the kinetic energy of the system accurately, it cannot represent the strain energy adequately. For instance, it is well known from Saint-Venant's theory for torsion that the cross-section will warp under torque, leading to a three-dimensional deformation state that generates a complex stress state. To overcome this problem, sectional stiffnesses are computed based on sophisticated mechanics of material theories that evaluate the complete state of deformation. These sectional stiffnesses are then used within the framework of an Euler-Bernoulli beam theory based on far simpler kinematic assumptions. While this approach works well for simple cross-sections made of homogeneous material, inaccurate predictions may result for realistic configurations, such as thin-walled sections, or sections comprising anisotropic materials. This paper presents a different approach to the problem. Based on a finite element discretization of the cross-section, an exact solution of the theory of three-dimensional elasticity is developed. The only approximation is that inherent to the finite element discretization. The proposed approach is based on the Hamiltonian formalism and leads to an expansion of the solution in terms of extremity and central solutions, as expected from Saint-Venant's principle.

1 Introduction

A beam is defined as a structure having one of its dimensions much larger than the other two. The generally curved axis of the beam is defined along that longer dimension and the cross-section slides along this axis. The cross-section's geometric and physical properties are assumed to vary smoothly along the beam's span. Numerous components found in flexible multibody systems are beam-like structures: linkages, transmission shafts, robotic arms, etc. Aeronautical structures such as aircraft wings or helicopter rotor blades are often treated as thin-walled beams.

The solid mechanics theory of beams, more commonly referred to simply as "beam theory," plays an important role in structural analysis because it provides designers with simple tools to analyze numerous structures [1]. The governing equations for beam structures are nonlinear partial differential equations, and the finite element method is often used to obtain approximate numerical solutions of these equations. Of course, the same finite element approach could also be used to

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model the same structures based on plate and shell, or even three-dimensional elasticity models, but at a far higher computation cost.

Several beam theories have been developed based on various assumptions, and lead to different levels of accuracy. One of the simplest and most useful of these theories is due to Euler who analyzed the elastic deformation of a slender beam. Euler-Bernoulli beam theory [1] is now commonly used in many civil, mechanical and aerospace applications, although shear deformable beam theories [2], often called "Timoshenko beams," have also found wide acceptance.

Most beam formulations used in multibody dynamics are based on kinematic assumptions; typically, the cross-section is assumed to remain rigid. Hence, its motion can be represented by six generalized coordinates, which are used to evaluate six sectional strains and six sectional velocities that define the strain and kinetic energy densities stored in the section, respectively. The now classical beam formulations of Simo *et al.* [3] or Géradin *et al.* [4] fall into that category and a comprehensive review of the many formulations used in multibody dynamics is given by Wasfy and Noor [5].

It is important to recognize the inconsistency of these classical formulations. While the rigid cross-section assumption is adequate to evaluate the kinetic energy, it cannot yield an accurate estimate of the strain energy. For instance, it is well known from Saint-Venant's theory that the torsional behavior of beams involves warping of their cross-sections. The torsional stiffness predicted by simplified theories based on rigid cross-sections could grossly over-estimate the actual sectional stifnesses obtained from Saint-Venant's theory. Similar remarks could be made concerning the shearing or even bending stiffness of beams, underlining the fundamental difficulties associated with the classical approaches.

To overcome these difficulties, researchers turn to an energy argument. Let J_{SV} be the torsional stiffness predicted by Saint-Venant's theory and $A_{SV} = 1/2\ J_{SV}\kappa^2$ the associated strain energy, where κ denotes the twist rate. Based on the rigid cross-section assumption, the corresponding quantities are J_{RC} and $A_{RC} = 1/2\ J_{RC}\kappa^2$, respectively, where $J_{SV} \leq J_{RC}$. A simple energy argument is then used: the strain energies based on the two assumptions become equal, *i.e.*, $A_{RC} = A_{SV}$, simply by selecting $J_{RC} = J_{SV}$.

A further inconsistency of the classical approach should be underlined. While beam theories based on the rigid cross-section assumption are used for flexible multibody systems undergoing large displacement and rotations, the energy equivalence argument presented above is based on Saint-Venant's theory, a solution of linear elasticity equations, which assume infinitesimal displacements and strains.

These inconsistencies have been observed by many authors over the last three decades and are exacerbated when dealing with complex build-up structures presenting solid or thin-walled cross-sections. Furthermore, laminated composite materials have found increased use in many flexible multibody systems such as wind turbine blades or aeronautical constructions leading to heterogeneous, highly anisotropic structures. In such cases, it is not clear how the simple energy equivalence argument presented earlier can be used reliably.

Giavotto *et. al.* [6] were the first to present a comprehensive solution of the problem based on linear elasticity theory. Their approach, based on an energy equivalence argument, leads to a two-dimensional analysis of the beam's cross-section using finite elements and yields the sectional stiffness characteristics in the form of a 6×6 sectional stiffness matrix. Furthermore, the three-dimensional strain field at all points of the cross-section can be recovered once the sectional strains are known. This work also identifies the two types of solutions present in beams: the central solutions and the extremity solutions, as should be expected from Saint-Venant's principle [7].

While the authors cited above focused on practical applications of engineering beam problems, far more theoretical approaches were also developed. For instance, Mielke [8, 9] investigated prismatic structures and identified the existence of a "central manifold," equivalent to the "central solutions" presented by Giavotto *et. al.* [6]. In a later paper [10], the same author developed a

Hamiltonian formulation to the problem.

At the same time, Zhong *et al.* [11, 12, 13] developed novel analytical techniques based on the Hamiltonian formalism for a broad class of linear elasticity problems. A Hamiltonian matrix characterizes the stiffness of the structure and its null eigenvalues give rise to polynomial solutions. Zhong further identified the fact the Hamiltonian matrix cannot be diagonalized, rather, it can be reduced the Jordan canonical form only. Finally, a similar approach was followed by Morandini *et al.* [14] who used numerical techniques to evaluate the Jordan form and associated generalized eigenvectors for cross-sections made of both isotropic and anisotropic materials.

For nonlinear problems, the decomposition of the beam problem into a linear, two-dimensional analysis over the cross-section, and a nonlinear, one-dimensional analysis along its span was first proposed by Berdichevsky [15]. Hodges [16] has reviewed many approaches to beam modeling; he points out that although the two-dimensional finite element analysis of the cross-section seems to be computationally expensive, it is, in fact, a preprocessing step that is performed once only.

A unified theory presenting both linear, two-dimensional analysis over the cross-section, and a nonlinear, one-dimensional analysis along the beam's span was further refined by Hodges and his co-workers [17, 18]. The nonlinear, one-dimensional analysis along the beam's span corresponds the geometrically exact beam theory developed earlier by Simo and co-workers [3]. Detailed developments of the nonlinear composite beam theory proposed by Hodges and his coworkers are found in his textbook [19].

In a first step towards the development of a comprehensive framework for modeling beams in multibody dynamics, the present paper presents a static solution of the problem, based on linear elasticity theory. Because the solution presented here is exact, it should provide a sound basis for generalization to the more challenging problems offered by flexible multibody systems.

2 Kinematics of the problem

Figure 1 depicts an initially straight beam of length L, with a cross-section of arbitrary shape and area $\mathcal{A}$. The volume of the beam is generated by sliding the cross-section along the reference line of the beam, which is defined by a straight line in space denoted $\mathcal{C}$. The beam's geometric and material properties are assumed to be uniform along its span. Curvilinear coordinate $s = \alpha_1$ measures length along $\mathcal{C}$. Point $\mathbf{B}$ is located at the intersection of the reference line with the plane of the cross-section. The unit tangent vector to line $\mathcal{C}$ is

$$\bar{t} = \frac{\partial \underline{r}_B}{\partial \alpha_1}, \tag{1}$$

where $\underline{r}_B$ is the position vector of point $\mathbf{B}$ with respect to the origin of the reference frame, $\mathcal{F} = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$.

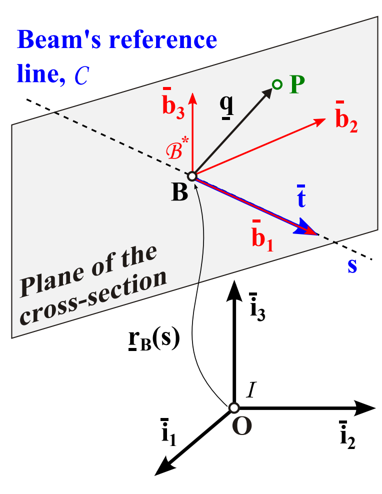

Figure 1: Configuration of a straight beam.

The plane of the cross-section is determined by two mutually orthogonal unit vectors, $\bar{b}_2$ and $\bar{b}_3$; orthonormal basis $\mathcal{B}^* = (\bar{b}_1, \bar{b}_2, \bar{b}_3)$ is then defined and unit tangent vector $\bar{t}$ is aligned with unit vector $\bar{b}_1$, as illustrated in fig. 1. A set of material coordinates that naturally represent the configuration of the beam is selected as follows: $\alpha_1 = s$ is the curvilinear variable that measure length along line $\mathcal{C}$, and α_2 and α_3 measure length along the directions of unit vectors $\bar{b}_2$ and $\bar{b}_3$, respectively.

It will be convenient to develop the model using a non-dimensional formulation. A representative dimension of the beam's cross-section, denoted a_r, will be used to normalize length and hence, $\bar{\alpha}_i =$

α_i/a_r, $i = 1, 2, 3$, are the non-dimensional coordinates of the problem and notation $(\cdot)' = a_r \mathrm{d}(\cdot)/\mathrm{d}\alpha_1$ indicates a derivative with respect to the non-dimensional spanwise variable $\bar{\alpha}_1$. The rotation tensor that brings basis $\mathcal{I}$ to basis $\mathcal{B}^*$ is denoted $\underline{\underline{R}}$.

2.1 Position and base vectors in the reference configuration

With these definitions, the position vector of an arbitrary material point of the beam becomes

$$\begin{aligned} \underline{r}(\alpha_1, \alpha_2, \alpha_3) =& \underline{r}_B(\alpha_1) + \alpha_2 \bar{b}_2 + \alpha_3 \bar{b}_3 \\ =& \underline{r}_B(\alpha_1) + \underline{q}(\alpha_2, \alpha_3), \end{aligned} \tag{2}$$

where vector $\underline{q} = \underline{r} - \underline{r}_B = \alpha_2 \bar{b}_2 + \alpha_3 \bar{b}_3$ defines the relative position of point $\mathbf{P}$ with respect to point $\mathbf{B}$. The base vectors associated with these material coordinates are $\underline{g}_1 = \partial \underline{r}/\partial \alpha_1 = \bar{b}_1$, $\underline{g}_2 = \partial \underline{r}/\partial \alpha_2 = \bar{b}_2$, and $\underline{g}_3 = \partial \underline{r}/\partial \alpha_3 = \bar{b}_3$. The components of these base vectors resolved in basis $\mathcal{B}^*$ are found to be

$$\underline{g}_1^* = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \underline{g}_2^* = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \underline{g}_3^* = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \tag{3}$$

Notation $(\cdot)^*$ indicates tensor components resolved in basis $\mathcal{B}^*$. Equations (3) now yield the components of the metric tensor for the reference configuration of the beam resolved in basis $\mathcal{B}^*$ as $\underline{\underline{g}}^* = \underline{\underline{I}}$, where $\underline{\underline{I}}$ is the identity matrix.

2.2 Position and base vectors in the deformed configuration

After deformation, the position vector of a material point becomes

$$\underline{R}(\alpha_1, \alpha_2, \alpha_3) = \underline{r} + \underline{u} = \underline{r} + u_i^* \bar{b}_i, \tag{4}$$

where $\underline{u}$ is the displacement vector and its components resolved in basis $\mathcal{B}^*$ are denoted u_i^*, *i.e.*, $\underline{u} = u_1^* \bar{b}_1 + u_2^* \bar{b}_2 + u_3^* \bar{b}_3$. The base vectors in the deformed configuration are now readily obtained as

$$\underline{G}_1 = \frac{\partial \underline{R}}{\partial \alpha_1} = \underline{g}_1 + \frac{\partial \bar{u}_i^*}{\partial \bar{\alpha}_1} \bar{b}_i, \tag{5a}$$

$$\underline{G}_2 = \frac{\partial \underline{R}}{\partial \alpha_2} = \underline{g}_2 + \frac{\partial \bar{u}_i^*}{\partial \bar{\alpha}_2} \bar{b}_i, \tag{5b}$$

$$\underline{G}_3 = \frac{\partial \underline{R}}{\partial \alpha_3} = \underline{g}_3 + \frac{\partial \bar{u}_i^*}{\partial \bar{\alpha}_3} \bar{b}_i. \tag{5c}$$

Resolving all vectors in basis $\mathcal{B}^*$ and using eqs. (5) and (3) yields the components of the base vectors of the deformed configuration as

$$\underline{G}_1^* = \begin{Bmatrix} 1 + \partial \bar{u}_1^*/\partial \bar{\alpha}_1 \\ \partial \bar{u}_2^*/\partial \bar{\alpha}_1 \\ \partial \bar{u}_3^*/\partial \bar{\alpha}_1 \end{Bmatrix}, \tag{6}$$

and

$$\underline{G}_2^* = \begin{Bmatrix} \partial \bar{u}_1^*/\partial \bar{\alpha}_2 \\ 1 + \partial \bar{u}_2^*/\partial \bar{\alpha}_2 \\ \partial \bar{u}_3^*/\partial \bar{\alpha}_2 \end{Bmatrix}, \quad \underline{G}_3^* = \begin{Bmatrix} \partial \bar{u}_1^*/\partial \bar{\alpha}_3 \\ \partial \bar{u}_2^*/\partial \bar{\alpha}_3 \\ 1 + \partial \bar{u}_3^*/\partial \bar{\alpha}_3 \end{Bmatrix}. \tag{7}$$

These expressions then yield the components of the deformation gradient tensor resolved in basis $\mathcal{B}^*$

$$\underline{\underline{F}}^* = \begin{bmatrix} 1 + \partial \bar{u}_1^*/\partial \bar{\alpha}_1 & \partial \bar{u}_1^*/\partial \bar{\alpha}_2 & \partial \bar{u}_1^*/\partial \bar{\alpha}_3 \\ \partial \bar{u}_2^*/\partial \bar{\alpha}_1 & 1 + \partial \bar{u}_2^*/\partial \bar{\alpha}_2 & \partial \bar{u}_2^*/\partial \bar{\alpha}_3 \\ \partial \bar{u}_3^*/\partial \bar{\alpha}_1 & \partial \bar{u}_3^*/\partial \bar{\alpha}_2 & 1 + \partial \bar{u}_3^*/\partial \bar{\alpha}_3 \end{bmatrix}. \tag{8}$$

2.3 Strain components

It is now assumed that the beam undergoes small displacements and rotations, and strain components remain very small at all times. Biot's strain tensor, defined as $\underline{\gamma}^* = (\underline{\underline{F}}^* + \underline{\underline{F}}^{*T})/2 - \underline{\underline{I}}$, is an adequate strain measure for the present small strain problem. The out-of-plane strain components, denoted $\underline{\gamma}_O^*$, and the in-plane components, denoted $\underline{\gamma}_I^*$, are

$$\underline{\gamma}_O^* = \begin{Bmatrix} \gamma_{11}^* \\ 2\gamma_{12}^* \\ 2\gamma_{13}^* \end{Bmatrix} = \underline{\bar{u}}^{*\prime} + \underline{\underline{\bar{D}}}_O \underline{\bar{u}}^*, \underline{\gamma}_I^* = \begin{Bmatrix} \gamma_{22}^* \\ \gamma_{33}^* \\ 2\gamma_{23}^* \end{Bmatrix} = \underline{\underline{\bar{D}}}_I \underline{\bar{u}}^*, \tag{9}$$

respectively. For future reference, the components of Biot's strain tensor are collected into a single array,

$$\underline{\gamma}^* = \begin{Bmatrix} \underline{\gamma}_O^* \\ \underline{\gamma}_I^* \end{Bmatrix} = \underline{\underline{\bar{A}}}\, \underline{\bar{u}}^{*\prime} + \underline{\underline{\bar{B}}}\, \underline{\bar{u}}^*. \tag{10}$$

In eqs. (9), the following differential operators were defined

$$\underline{\underline{\bar{D}}}_O = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{\partial}{\partial \bar{\alpha}_2} & 0 & 0 \\ \dfrac{\partial}{\partial \bar{\alpha}_3} & 0 & 0 \end{bmatrix}, \quad \underline{\underline{\bar{D}}}_I = \begin{bmatrix} 0 & \dfrac{\partial}{\partial \bar{\alpha}_2} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial \bar{\alpha}_3} \\ 0 & \dfrac{\partial}{\partial \alpha_3} & \dfrac{\partial}{\partial \bar{\alpha}_2} \end{bmatrix}. \tag{11}$$

In eq. (10), the following differential operators were defined

$$\underline{\underline{\bar{A}}} = \begin{bmatrix} \underline{\underline{I}} \\ \underline{\underline{0}} \end{bmatrix}, \quad \underline{\underline{\bar{B}}} = \begin{bmatrix} \underline{\underline{\bar{D}}}_O \\ \underline{\underline{\bar{D}}}_I \end{bmatrix}. \tag{12}$$

2.4 Semi-discretization of the displacement field

Beam theory is characterized by one-dimensional, ordinary differential equations governing the displacement field assumed to be a function of the axial variable, $\bar{\alpha}_1$, only. In the above paragraphs, the displacement field has been treated as a general vector field depending on three independent variables, $\bar{\alpha}_1$, $\bar{\alpha}_2$, and $\bar{\alpha}_3$. To obtain a one-dimensional formulation, the following semi-discretization of the displacement field is performed,

$$\underline{\bar{u}}^*(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) = \underline{\underline{N}}(\bar{\alpha}_2, \bar{\alpha}_3)\underline{\hat{u}}(\bar{\alpha}_1), \tag{13}$$

where matrix $\underline{\underline{N}}(\bar{\alpha}_2, \bar{\alpha}_3)$ stores the two-dimensional shape functions used in the discretization and array $\underline{\hat{u}}(\bar{\alpha}_1)$ the nodal values of the non-dimensional displacement field.

Equation (13) expresses the discretization of the displacement field and is very similar to the standard discretization procedure used in the finite element method [20, 21], although important differences exist. In typical finite element approximations, the beam would be treated as a three-dimensional body and eq. (13) would be written as $\underline{u}^*(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) = \underline{\underline{N}}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3)\underline{\hat{u}}$, where matrix $\underline{\underline{N}}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3)$ would store three-dimensional shape functions and $\underline{\hat{u}}$ the values of the displacement components at the nodes of the discretization. In this case, the finite element mesh would cover the beam's cross-section, but would also extend along its span. Array $\underline{\hat{u}}$ would store the nodal displacement components at all nodes over the cross-section and along the beam's span; this corresponds to a full discretization of the problem. Notation $\hat{(\cdot)}$ indicates nodal quantities of the discretized model.

Equation (13) corresponds to a *semi-discretization* of the problem and was first proposed by Borri *et al.* [6]. The finite element mesh extends over the beam's cross-section only and the nodal values of the displacement components remain functions of the axial variable, $\bar{\alpha}_1$. Consequently, the shape functions depend on the variables describing the geometry of the cross-section, $\bar{\alpha}_2$ and $\bar{\alpha}_3$, only, *i.e.*, $\underline{\underline{N}} = \underline{\underline{N}}(\bar{\alpha}_2, \bar{\alpha}_3)$. The dependency of the nodal displacement components on the axial variable is explicitly stated as $\underline{\hat{u}} = \underline{\hat{u}}(\bar{\alpha}_1)$. The semi-discretization procedure leads to a numerical treatment of the solution for variables $\bar{\alpha}_2$ and $\bar{\alpha}_3$ whereas the dependency of the solution on variable $\bar{\alpha}_1$ is treated analytically.

Introducing this discretization into eq. (10) yields the components of Biot's strain tensor as

$$\underline{\gamma}^* = \underline{\underline{\bar{A}}}\,\underline{\underline{N}}\,\underline{\hat{u}}' + \underline{\underline{\bar{B}}}\,\underline{\underline{N}}\,\underline{\hat{u}}. \tag{14}$$

3 Governing equations

Because much of the work presented here relies on Hamilton's formulation, the beam's governing equations will be derived using Hamilton's canonical approach. The more classical method based on the principle of virtual work could also be used and leads to identical results. The components of Biot's strain tensor, $\underline{\gamma}^*$, and of the convected Cauchy stress tensor, $\underline{\tau}^*$, are energetically conjugate quantities for small strain problems and yield the strain energy of the system.

3.1 Strain energy expression

The strain energy, which is the Lagrangian of the problem, stored in an infinitesimal slice of the beam of span ds is

$$L = \frac{1}{2}\int_v \underline{\gamma}^{*T}\underline{\tau}^* \;\mathrm{d}v = \frac{1}{2}\int_{\mathcal{A}} \underline{\gamma}^{*T}\underline{\underline{\mathcal{D}}}^*\,\underline{\gamma}^* \;\mathrm{d}\mathcal{A}\mathrm{d}s. \tag{15}$$

In the second equality, the components of the convected Cauchy stress tensor were related to the Biot strain components using the constitutive laws for linearly elastic materials,

$$\underline{\tau}^* = \underline{\underline{\mathcal{D}}}^*\underline{\gamma}^*, \tag{16}$$

where $\underline{\underline{\mathcal{D}}}^*$ is a 6×6 stiffness matrix for the material, resolved in material basis $\mathcal{B}^*$. The non-dimensional form of the constitutive laws is $\underline{\bar{\tau}}^* = \underline{\underline{\bar{\mathcal{D}}}}^*\underline{\gamma}^*$, where $\underline{\bar{\tau}}^* = \underline{\tau}^*/E_r$, $\underline{\underline{\bar{\mathcal{D}}}}^* = \underline{\underline{\mathcal{D}}}^*/E_r$, and E_r is a representative value of Young's modulus.

Introducing the discretized components of Biot's strain tensor given by eq. (14), the strain energy becomes

$$2\bar{L} = \underline{\hat{u}}'^T\left(\underline{\underline{\bar{M}}}\,\underline{\hat{u}}' + \underline{\underline{\bar{C}}}^T\underline{\hat{u}}\right) + \underline{\hat{u}}^T\left(\underline{\underline{\bar{C}}}\,\underline{\hat{u}}' + \underline{\underline{\bar{E}}}\,\underline{\hat{u}}\right), \tag{17}$$

where $\bar{L} = L/(a_r^2 E_r)$ is the non-dimensional strain energy and matrices $\underline{\underline{\bar{M}}}$, $\underline{\underline{\bar{C}}}$, and $\underline{\underline{\bar{E}}}$ are defined as

$$\underline{\underline{\bar{M}}} = \int_{\bar{\mathcal{A}}} (\underline{\underline{\bar{A}}}\,\underline{\underline{N}})^T\underline{\underline{\bar{\mathcal{D}}}}^*(\underline{\underline{\bar{A}}}\,\underline{\underline{N}}) \;\mathrm{d}\bar{\mathcal{A}}, \tag{18a}$$

$$\underline{\underline{\bar{C}}} = \int_{\bar{\mathcal{A}}} (\underline{\underline{\bar{B}}}\,\underline{\underline{N}})^T\underline{\underline{\bar{\mathcal{D}}}}^*(\underline{\underline{\bar{A}}}\,\underline{\underline{N}}) \;\mathrm{d}\bar{\mathcal{A}}, \tag{18b}$$

$$\underline{\underline{\bar{E}}} = \int_{\bar{\mathcal{A}}} (\underline{\underline{\bar{B}}}\,\underline{\underline{N}})^T\underline{\underline{\bar{\mathcal{D}}}}^*(\underline{\underline{\bar{B}}}\,\underline{\underline{N}}) \;\mathrm{d}\bar{\mathcal{A}}, \tag{18c}$$

where $\bar{\mathcal{A}} = \mathcal{A}/a_r^2$ is the non-dimensional area of the cross-section. Assuming that the beam's cross-section has been discretized using N nodes, the total number of degrees of freedom is $n = 3N$ and matrices $\underline{\underline{\bar{M}}}$, $\underline{\underline{\bar{C}}}$, and $\underline{\underline{\bar{E}}}$ are of size $n \times n$.

A set of dual variables, denoted $\underline{\hat{P}}$, is introduced

$$\underline{\hat{P}} = \frac{\partial \bar{L}}{\partial \underline{\hat{u}}'} = \underline{\underline{\bar{M}}}\,\underline{\hat{u}}' + \underline{\underline{\bar{C}}}^T\,\underline{\hat{u}}, \tag{19}$$

which can be interpreted as non-dimensional nodal forces. Next, the non-dimensional Hamiltonian of the system, denoted $\bar{H}$, is defined via Legendre's transformation [22] as $\bar{H} = \underline{\hat{P}}^T\underline{\hat{u}}' - \bar{L}$ and tedious algebra reveals that

$$\bar{H} = \frac{1}{2}\underline{\hat{\mathcal{X}}}^T\,\underline{\underline{\bar{\mathcal{F}}}}\,\underline{\hat{\mathcal{X}}}, \tag{20}$$

where array $\underline{\hat{\mathcal{X}}}$ stores the nodal displacements and forces

$$\underline{\hat{\mathcal{X}}} = \left\{ \begin{matrix} \underline{\hat{u}} \\ \underline{\hat{P}} \end{matrix} \right\}, \tag{21}$$

and matrix $\underline{\underline{\bar{\mathcal{F}}}}$ is defined as

$$\underline{\underline{\bar{\mathcal{F}}}} = \begin{bmatrix} \underline{\underline{\bar{C}}}\,\underline{\underline{\bar{M}}}^{-1}\underline{\underline{\bar{C}}}^T - \underline{\underline{\bar{E}}} & -\underline{\underline{\bar{C}}}\,\underline{\underline{\bar{M}}}^{-1} \\ -(\underline{\underline{\bar{C}}}\,\underline{\underline{\bar{M}}}^{-1})^T & \underline{\underline{\bar{M}}}^{-1} \end{bmatrix}. \tag{22}$$

Hamilton's canonical approach provides two sets of equations. The first set, $\underline{\hat{u}}' = \partial\bar{H}/\partial\underline{\hat{P}}$, is identical to eqs. (19), *i.e.*, merely defines the nodal forces. The second set, $\underline{\hat{P}}' = -\partial\bar{H}/\partial\underline{\hat{u}}$, provides the governing equations of the problem. The combination of the two sets defines $2n$ first-order, ordinary differential equations with constant coefficients

$$\underline{\hat{\mathcal{X}}}' = \underline{\underline{\bar{\mathcal{H}}}}\,\underline{\hat{\mathcal{X}}}, \tag{23}$$

where matrix $\underline{\underline{\bar{\mathcal{H}}}}$, of size $2n \times 2n$, is defined as

$$\underline{\underline{\bar{\mathcal{H}}}} = \begin{bmatrix} -\underline{\underline{\bar{M}}}^{-1}\underline{\underline{\bar{C}}}^T & \underline{\underline{\bar{M}}}^{-1} \\ \underline{\underline{\bar{E}}} - \underline{\underline{\bar{C}}}\,\underline{\underline{\bar{M}}}^{-1}\underline{\underline{\bar{C}}}^T & \underline{\underline{\bar{C}}}\,\underline{\underline{\bar{M}}}^{-1} \end{bmatrix} = \underline{\underline{\mathcal{J}}}\,\underline{\underline{\bar{\mathcal{F}}}}. \tag{24}$$

In view of definition (67), it is verified easily that this matrix is a Hamiltonian matrix and matrix $\underline{\underline{\mathcal{J}}}$ is defined by eq. (68).

It is interesting to note that $\bar{H}' = (\partial\bar{H}/\partial\underline{\hat{u}})\underline{\hat{u}}' + (\partial\bar{H}/\partial\underline{\hat{P}})\underline{\hat{P}}' = 0$, where the second equality follows from the canonical equations. Clearly, the Hamiltonian remains constant along the beam's span, and hence, can be interpreted as the strain energy density. It the follows that matrix $\underline{\underline{\bar{\mathcal{F}}}}$ defined by eq. (22) is a positive-definite matrix. Note that the Hamiltonian can also be written as $\bar{H} = (\underline{\hat{u}}'^T\underline{\underline{\bar{M}}}\,\underline{\hat{u}}' - \underline{\hat{u}}^T\underline{\underline{\bar{E}}}\,\underline{\hat{u}})/2$, which implies that the difference between the two positive quantities, $\underline{\hat{u}}'^T\underline{\underline{\bar{M}}}\,\underline{\hat{u}}'$ and $\underline{\hat{u}}^T\underline{\underline{\bar{E}}}\,\underline{\hat{u}}$, must remain constant along the beam's span.

3.2 Rigid-body motion

Important information about the behavior of the system can be gained by considering a rigid-body displacement field written as $\underline{u} = \underline{u}_R - \tilde{q}\underline{\phi}_R$, where $\underline{u}_R$ are the components of a rigid-body translation and $\underline{\phi}_R$ those of an infinitesimal rigid-body rotation. For the cross-section of the beam located at curvilinear variable $\alpha_1 + \mathrm{d}\alpha_1$, the corresponding rigid-body displacement is $\underline{u}(\alpha_1 + \mathrm{d}\alpha_1) = \underline{u}_R - (\tilde{q} + \mathrm{d}\tilde{r})\underline{\phi}_R$; it then follows that the non-dimensional derivative of the rigid-body motion with respect to $\bar{\alpha}_1$ is $\underline{\bar{u}}' = -\tilde{t}\underline{\phi}_R$, where $\bar{t}$ is the unit tangent vector to line $\mathcal{C}$ defined by eq. (1).

When the components of the rigid-body motion are expressed in basis $\mathcal{B}^*$, the corresponding derivative is $\underline{\bar{u}}^{*\prime} = (\underline{\underline{R}}^T\underline{\bar{u}})' = \underline{\underline{R}}\,\underline{\bar{u}}' = \tilde{t}^{*T}\underline{\phi}^*_R$ and proceeding in a similar manner for the components of the infinitesimal rotation yields $\underline{\phi}_R^{*\prime} = \underline{0}$. Recasting these results in a matrix format yields $\underline{\mathcal{U}}_R^{*\prime} = -\tilde{\bar{\mathcal{K}}}^* \underline{\mathcal{U}}_R^*$. For convenience, the following non-dimensional rigid-body motion array is defined

$$\underline{\bar{\mathcal{U}}}_R^* = \begin{Bmatrix} \underline{\bar{u}}_R^* \\ \underline{\phi}_R^* \end{Bmatrix}, \tag{25}$$

together with the following 6×6 non-dimensional curvature tensor

$$\tilde{\bar{\mathcal{K}}}^* = \begin{bmatrix} \underline{\underline{0}} & \tilde{t}^* \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix}, \tag{26}$$

called the "generalized vector-product tensor."

At a specific point of the cross-section, the non-dimensional components of the rigid-body motion become

$$\begin{aligned} \begin{Bmatrix} \bar{u}_1^* \\ \bar{u}_2^* \\ \bar{u}_3^* \end{Bmatrix} &= \underline{\bar{u}}_R^* - \tilde{\bar{q}}^* \underline{\phi}_R^* \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & \bar{\alpha}_3 & -\bar{\alpha}_2 \\ 0 & 1 & 0 & -\bar{\alpha}_3 & 0 & 0 \\ 0 & 0 & 1 & \bar{\alpha}_2 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \underline{\bar{u}}_R^* \\ \underline{\phi}_R^* \end{Bmatrix} = \underline{\underline{\bar{z}}} \underline{\bar{\mathcal{U}}}_R^*. \end{aligned} \tag{27}$$

Collecting this information for all the nodes of the discretization then yields $\underline{\hat{u}} = \underline{\underline{\bar{Z}}} \underline{\bar{\mathcal{U}}}_R^*$ and $\underline{\hat{u}}' = -\underline{\underline{\bar{Z}}}\, \tilde{\bar{\mathcal{K}}}^* \underline{\bar{\mathcal{U}}}_R^*$, where matrix $\underline{\underline{\bar{Z}}}$ stacks the rows of matrix $\underline{\underline{\bar{z}}}$ for each of the nodes.

Note that the first three components of array $\underline{\underline{\bar{Z}}}^T \underline{\hat{P}}$ corresponds to the summation of the nodal forces over the beam's cross-section, whereas the last three components correspond to the sums of the moments of the same forces, leading to

$$\underline{\bar{\mathcal{F}}}^* = \begin{Bmatrix} \underline{\bar{f}}^* \\ \underline{\bar{m}}^* \end{Bmatrix} = \underline{\underline{\bar{Z}}}^T \underline{\hat{P}}. \tag{28}$$

where $\underline{\bar{f}}^{*T} = \left\{ \bar{N}_1^*, \bar{N}_2^*, \bar{N}_3^* \right\}$ and $\underline{\bar{m}}^{*T} = \left\{ \bar{M}_1^*, \bar{M}_2^*, \bar{M}_3^* \right\}$ are the components of the non-dimensional stress resultant forces and moments, respectively, resolved in basis $\mathcal{B}^*$. The sectional forces include the axial force, N_1^*, and the two transverse shear forces, N_2^* and N_3^*, the moments include the twisting moment, M_1^*, and the two bending moments, M_2^* and M_3^*.

For the present rigid-body motions, eq. (17) gives the strain energy of the system as

$$\begin{aligned} \bar{L} =& \underline{\hat{u}}_R^{T\prime} \left(\underline{\underline{\bar{M}}}\, \underline{\hat{u}}_R' + \underline{\underline{\bar{C}}}^T \underline{\hat{u}}_R \right) + \underline{\hat{u}}_R^T \left(\underline{\underline{\bar{C}}}\, \underline{\hat{u}}_R' + \underline{\underline{\bar{E}}}\, \underline{\hat{u}}_R \right) \\ =& \underline{\hat{u}}_R^{T\prime} \left[-\underline{\underline{\bar{M}}}\, \underline{\underline{\bar{Z}}}\, \tilde{\bar{\mathcal{K}}}^* + \underline{\underline{\bar{C}}}^T \underline{\underline{\bar{Z}}} \right] \underline{\bar{\mathcal{U}}}_R^* \\ &+ \underline{\hat{u}}_R^T \left[-\underline{\underline{\bar{C}}}\, \underline{\underline{\bar{Z}}}\, \tilde{\bar{\mathcal{K}}}^* + \underline{\underline{\bar{E}}}\, \underline{\underline{\bar{Z}}} \right] \underline{\bar{\mathcal{U}}}_R^*. \end{aligned} \tag{29}$$

Because the strain energy must vanish for all rigid-body motions, the bracketed terms must vanish, yielding the following matrix identities

$$\underline{\underline{\bar{C}}}^T \underline{\underline{\bar{Z}}} = \underline{\underline{\bar{M}}}\, \underline{\underline{\bar{Z}}}\, \tilde{\bar{\mathcal{K}}}^*, \tag{30a}$$

$$\underline{\underline{\bar{E}}}\, \underline{\underline{\bar{Z}}} = \underline{\underline{\bar{C}}}\, \underline{\underline{\bar{Z}}}\, \tilde{\bar{\mathcal{K}}}^*. \tag{30b}$$

3.3 Characteristics of the governing equations

Because the governing equations of the problem are first-order, ordinary differential equations with constant coefficients, the solutions are expected to be in the form of exponential functions, $\underline{\hat{\mathcal{X}}}(\bar{\alpha}_1) = \underline{\bar{\mathcal{X}}}_i \exp(\bar{\lambda}_i \bar{\alpha}_1)$, where $\bar{\lambda}_i$ denotes the non-dimensional characteristic exponents of the system, which are the eigenvalues of Hamiltonian matrix $\underline{\underline{\bar{\mathcal{H}}}}$. As shown in section A.1, the eigenvalues of Hamiltonian matrices occur in pairs of opposite signs, $\pm\bar{\lambda}_i$, leading to solutions of the form $\underline{\hat{\mathcal{X}}}(\bar{\alpha}_1) = \underline{\bar{\mathcal{X}}}_+ \exp(+\bar{\lambda}_i \bar{\alpha}_1) + \underline{\bar{\mathcal{X}}}_- \exp(-\bar{\lambda}_i \bar{\alpha}_1)$. Based on simple physical arguments, the exponential solution $\exp(+\bar{\lambda}_i)$ makes little sense if $\Re(\bar{\lambda}_i) > 0$. Consequently, the solutions associated with characteristic exponents of opposite signs correspond to exponentially decaying solutions away from the beam's ends. These solutions, which are ignored in engineering beam theories, are known as the "extremity solutions" of the problem and are to be expected as a consequence of Saint-Venant's principle [7, 1]. Extremity solutions can be viewed as boundary layer solutions.

Due to the presence of rigid-body modes, Hamiltonian matrix $\underline{\underline{\bar{\mathcal{H}}}}$ also possesses a zero eigenvalue. For straight beams, it will be shown that the zero eigenvalue has a multiplicity of twelve: the first six eigenvectors are the beam's rigid-body modes, and the next six provide the beam's fundamental deformation modes. Because the multiplicity of this eigenvalue is larger than one, polynomial type solutions are to be expected, which are non-vanishing over the beam's entire span. Consequently, these solutions are known as the "central solutions," as opposed to the extremity solutions that are confined to the beam's ends.

The terms "central" and "extremity" solutions were coined by Borri *et al.* [6, 23] and are illustrated in fig. 2, which shows that an arbitrary stress distribution applied over the cross-section can be decomposed unequivocally into six stress resultants, the three sectional forces and moments, and a self-equilibrating stress distribution, *i.e.*, a stress distribution presenting no net resultant. The sectional stress resultants excite the central solutions only, which are polynomial solutions propagating over the entire span of the beam. The self-equilibrating stress distribution excites the extremity solutions only, which are exponentially decaying solutions.

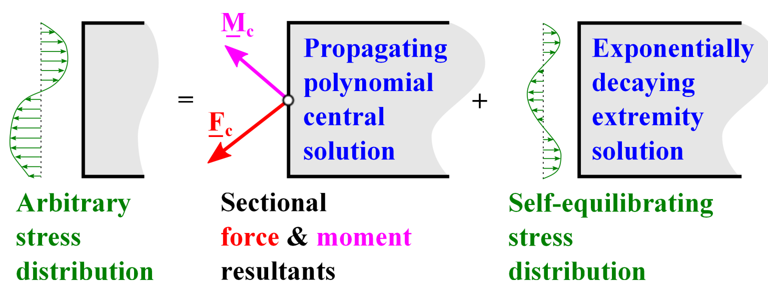

Figure 2: Illustration of Saint-Venant's principle.

Clearly, the determination of the central solutions is important, because they affect the beam's entire span as opposed to the extremity solutions that affect its ends only. Furthermore, the decomposition of the solution into its central and extremity components is a consequence of Saint-Venant's principle, which states [7, 1]: "If self-equilibrating loads are applied to a body over an area of characteristic dimension δ, the internal stresses resulting from these loads are only significant over a portion of the body of approximate characteristic dimension δ." The extremity solutions penetrate a distance δ into the elastic body because their amplitudes decay exponentially.

System (23) could be tackled numerically using standard tools for the solution of boundary value problems, but such approach would not distinguish between the central and extremity solutions. This paper uses a different approach that involves a coordinate transformation of the following form

$$\underline{\hat{\mathcal{X}}} = \underline{\underline{\mathcal{S}}}\,\underline{\mathcal{Q}}, \tag{31}$$

where $\underline{\underline{\mathcal{S}}}$ is a yet undetermined symplectic matrix of size $2n \times 2n$ and $\underline{\mathcal{Q}}$ an array of $2n$ modal coordinates. Introducing this coordinate transformation into eq. (23) then yields

$$\underline{\mathcal{Q}}' = \underline{\underline{\bar{\mathcal{H}}}}_r\,\underline{\mathcal{Q}}, \tag{32}$$

where $\underline{\underline{\bar{\mathcal{H}}}}_r = \underline{\underline{\mathcal{S}}}^{-1}\underline{\underline{\bar{\mathcal{H}}}}\,\underline{\underline{\mathcal{S}}}$ is still a Hamiltonian matrix, as implied by property (77). Through a careful selection of the symplectic transformation matrix $\underline{\underline{\mathcal{S}}}$, the reduced Hamiltonian matrix, $\underline{\underline{\bar{\mathcal{H}}}}_r$, will become sparse, enabling closed-form solution of eqs. (32) and the explicit separation of the solution into its central and extremity components.

The approach outlined in the previous paragraph is very classical and appears straightforward but is, in fact, fraught with difficulties. It mirrors the approach used to solve vibration problems, where the mass and stiffness matrices are diagonalized through a change of coordinates characterized by an orthogonal matrix, whose existence is guaranteed because both mass and stiffness matrices are symmetric and positive-definite. In the present case, the coordinate transformation is characterized by a symplectic matrix, which preserves the Hamiltonian nature of the problem. Unfortunately, because the Hamiltonian is neither symmetric nor positive-definite, it cannot always be transformed to a diagonal form. In most elasticity problems, the Hamiltonian possesses zero eigenvalues of high multiplicity and the associated null space is not spanned by orthogonal vectors. Rather, as shown by Zhong and coworkers [24, 13], the zero eigenvalue is associated with eigenvectors and generalized eigenvectors that reduce the Hamiltonian to Jordan canonical form. For the problem at hand, the multiplicity of the zero eigenvalue of the Hamiltonian is twelve and a Jordan canonical form results. Without recognizing this fact, it is not possible to obtain the correct analytical solution of the problem [25].

The existence of the zero eigenvalue is easy to prove. Indeed, identities (30) can be recast as

$$\underline{\underline{\bar{\mathcal{H}}}}\,\underline{\underline{\mathcal{Z}}} = -\underline{\underline{\mathcal{Z}}}\,\tilde{\bar{\mathcal{K}}}^*, \tag{33}$$

where matrix $\underline{\underline{\mathcal{Z}}}$, of size $2n \times 6$, is defined as

$$\underline{\underline{\mathcal{Z}}} = \begin{bmatrix} \underline{\underline{\bar{Z}}} \\ \underline{\underline{0}} \end{bmatrix}. \tag{34}$$

For straight beams, four columns of matrix $\tilde{\bar{\mathcal{K}}}^*$ vanish, which implies that Hamiltonian matrix $\underline{\underline{\bar{\mathcal{H}}}}$ is at least four times singular and the corresponding columns of matrix $\underline{\underline{\mathcal{Z}}}$ store the associated eigenvectors, which are, as expected, the beam's rigid-body motions.

4 Reduction of the Hamiltonian matrix

As indicated earlier, the Hamiltonian of the system will be reduced to a nearly diagonal form through the use of a coordinate transformations characterized by a symplectic matrix. For convenience, this process is broken into two steps. In section 4.1, it is assumed that the Hamiltonian possesses non-vanishing eigenvalues only and the corresponding extremity solutions provide the desired symplectic matrix. In section 4.2, the more difficult case of the vanishing eigenvalues is addressed and the associated central solutions lead to the desired transformation.

4.1 Extremity solutions

The extremity solutions are associated with the eigenvalues of Hamiltonian matrix $\underline{\underline{\bar{\mathcal{H}}}}$ with a non-vanishing real part, $\Re(\lambda_i) \neq 0$. Numerical tools are used to extract these eigenvalues and associated eigenvectors. As proved in section A.1, these eigenvalues occur in pairs of opposite sign, $\pm\lambda_i$, where $+\lambda_i$ indicates the eigenvalue with a positive real part. Because the Hamiltonian is a real but unsymmetric matrix, complex eigenvalues are possible and will occur in pairs of complex conjugate roots, $\pm\lambda_i$ and $\pm\lambda_i^\dagger$, where notation $(\cdot)^\dagger$ indicates a complex conjugate number. The following matrix is constructed with all the eigenvectors of the Hamiltonian

$$\underline{\underline{\mathcal{U}}}_e = \left[\underline{\mathcal{U}}_{-\lambda_1}, \cdots, \underline{\mathcal{U}}_{-\lambda_n}, \underline{\mathcal{U}}_{+\lambda_1}, \cdots, \underline{\mathcal{U}}_{+\lambda_n}\right]. \tag{35}$$

Equation (72) now expands to $\underline{\underline{\mathcal{U}}}_e^T \underline{\underline{\mathcal{J}}}\, \underline{\underline{\mathcal{U}}}_e = \underline{\underline{\mathcal{J}}}$, which, in view of eq. (74), implies that $\underline{\underline{\mathcal{U}}}_e$ is a symplectic matrix. Because the inverse of a symplectic matrix is given by eq. (75), it follows that $\underline{\underline{\mathcal{U}}}_e^{-1} \underline{\underline{\bar{\mathcal{H}}}}\, \underline{\underline{\mathcal{U}}}_e = \underline{\underline{\mathcal{J}}}\, \underline{\underline{\mathcal{U}}}_e^T \underline{\underline{\mathcal{J}}}\, \underline{\underline{\bar{\mathcal{H}}}}\, \underline{\underline{\mathcal{U}}}_e$ and introducing eq. (73) now leads to

$$\underline{\underline{\mathcal{U}}}_e^{-1} \underline{\underline{\bar{\mathcal{H}}}}\, \underline{\underline{\mathcal{U}}}_e = \begin{bmatrix} \mathrm{diag}(-\lambda_\alpha) & \underline{\underline{0}} \\ \underline{\underline{0}} & \mathrm{diag}(+\lambda_\alpha) \end{bmatrix} = \underline{\underline{\bar{\mathcal{H}}}}_e. \tag{36}$$

For the extremity solutions, the coordinate transformation expressed by eq. (31) becomes $\underline{\hat{\mathcal{X}}} = \underline{\underline{\mathcal{U}}}_e \underline{\mathcal{Q}}_e$. It involves symplectic matrix $\underline{\underline{\mathcal{U}}}_e$ and modal coordinates $\underline{\mathcal{Q}} = \underline{\mathcal{Q}}_e$, which can be interpreted as the amplitudes of the extremity solution modes. As expected, the right-hand side of eq. (36) is a Hamiltonian matrix; it is also diagonal.

Consider the extremity solutions written as $\underline{\hat{\mathcal{X}}}_e = \underline{\underline{\mathcal{U}}}_e \underline{\mathcal{Q}}_e$; pre-multiplication by $\underline{\underline{\mathcal{Z}}}^T \underline{\underline{\mathcal{J}}}$ then yields $\underline{\underline{\mathcal{Z}}}^T \underline{\underline{\mathcal{J}}}\, \underline{\hat{\mathcal{X}}}_e = \underline{\underline{\mathcal{Z}}}^T \underline{\underline{\mathcal{J}}}\, \underline{\underline{\mathcal{U}}}_e \underline{\mathcal{Q}}_e = \underline{0}$, where the second equality follows from the fact that $\underline{\underline{\mathcal{Z}}}^T \underline{\underline{\mathcal{J}}}\, \underline{\underline{\mathcal{U}}}_e = \underline{0}$; indeed, the columns of matrix $\underline{\underline{\mathcal{U}}}_e$ store the eigenvectors associated eigenvalues $\Re(\lambda_i) \neq 0$ and the columns of matrix $\underline{\underline{\mathcal{Z}}}$ store the eigenvectors and generalized eigenvectors associated with the null eigenvalue, resulting in their symplectic orthogonality. Partitioning the solution array as $\underline{\hat{\mathcal{X}}}_e^T = \{\underline{\bar{u}}_e^T, \underline{\hat{P}}_e^T\}$ results in $\underline{\mathcal{F}}_e^* = \underline{\underline{Z}}^T \underline{\hat{P}}_e = \underline{\underline{\mathcal{Z}}}^T \underline{\underline{\mathcal{J}}}\, \underline{\hat{\mathcal{X}}}_e = \underline{0}$. In other words, the extremity solutions involve nodal force distributions that present no net resultant: they are self equilibrating, as expected from Saint-Venant's principle.

4.2 Central solutions

As was done for the extremity solutions, the central solutions are obtained from a coordinate transformation of the following type

$$\underline{\hat{\mathcal{X}}}_c = \underline{\underline{\mathcal{U}}}_c \underline{Q}_c, \tag{37}$$

where array $\underline{\hat{\mathcal{X}}}_c$ stores the nodal displacements and forces associated with the central solution

$$\underline{\hat{\mathcal{X}}}_c = \begin{Bmatrix} \underline{\hat{u}}_c \\ \underline{\hat{P}}_c \end{Bmatrix}, \quad \underline{Q}_c = \begin{Bmatrix} \underline{\bar{\mathcal{U}}}_c^* \\ \underline{\bar{\mathcal{F}}}_c^* \end{Bmatrix}, \tag{38}$$

and array $\underline{Q}_c$ the generalized coordinates, $\underline{\bar{\mathcal{U}}}_c^*$ and $\underline{\bar{\mathcal{F}}}_c^*$, which will be identified as the average sectional displacements and stress resultants, respectively. The coordinate transformation is defined by matrix $\underline{\underline{\mathcal{U}}}_c$, which is of the following form,

$$\underline{\underline{\mathcal{U}}}_c = \begin{bmatrix} \underline{\underline{\bar{Z}}} & \underline{\underline{\bar{W}}} \\ \underline{\underline{0}} & \underline{\underline{\bar{Y}}} \end{bmatrix}, \tag{39}$$

where matrices $\underline{\underline{\bar{W}}}$ and $\underline{\underline{\bar{Y}}}$ are of size $n \times 6$ and yet undetermined. Equation (39) implies that the nodal displacements of the central solutions can be expressed as $\underline{\hat{u}}_c = \underline{\underline{\bar{Z}}}\, \underline{\bar{\mathcal{U}}}_c^* + \underline{\underline{\bar{W}}}\, \underline{\bar{\mathcal{F}}}_c^*$, where the first term accounts for rigid-body motions and the second for warping; the columns of matrix $\underline{\underline{\bar{W}}}$ represent the warping induced by unit sectional stress resultants. The nodal forces are expressed as $\underline{\hat{P}}_c = \underline{\underline{\bar{Y}}}\, \underline{\bar{\mathcal{F}}}_c^*$; the columns of matrix $\underline{\underline{\bar{Y}}}$ represent the nodal forces induced by unit sectional stress resultants.

The work done by the nodal forces distributed over the beam's cross-section is $W = \underline{\hat{P}}_c^T \underline{\hat{u}}_c = \underline{\bar{\mathcal{F}}}_c^{*T} \underline{\underline{\bar{Y}}}^T (\underline{\underline{\bar{Z}}}\, \underline{\bar{\mathcal{U}}}_c^* + \underline{\underline{\bar{W}}}\, \underline{\bar{\mathcal{F}}}_c^*)$. If the following conditions are satisfied,

$$\underline{\underline{\bar{Z}}}^T \underline{\underline{\bar{Y}}} = \underline{\underline{I}}, \tag{40a}$$

$$\underline{\underline{\bar{W}}}^T \underline{\underline{\bar{Y}}} = \underline{\underline{0}}, \tag{40b}$$

this work reduces to

$$W = \underline{\hat{P}}_c^T \underline{\hat{u}}_c = \underline{\bar{\mathcal{F}}}_c^{*T} \underline{\bar{\mathcal{U}}}_c^*. \tag{41}$$

This implies that the generalized coordinates are energetically conjugated because their product yields the work done by the distributed nodal forces exactly. It is verified easily that conditions (40) are sufficient to prove that matrix $\underline{\underline{\mathcal{U}}}_c$ defining the coordinate transformation (37) is symplectic, as desired.

According to Clapeyron's theorem, $\underline{\hat{P}}_c^T \underline{\hat{u}}_c$ equals twice the strain energy stored in the beam if the nodal forces, $\underline{\hat{P}}_c$, are applied over the cross-section; similarly, $\underline{\bar{\mathcal{F}}}_c^{*T} \underline{\bar{\mathcal{U}}}_c^*$ equals twice the corresponding quantity if the six stress resultants, $\underline{\bar{\mathcal{F}}}_c^*$, are applied. Equation (41) can be interpreted as an energy equivalence statement: the strain energy resulting from the application of the nodal forces of the central solution equals that associated with the application of the six stress resultants. This fundamental equality allows the dimensional reduction of the problem.

Pre-multiplying eq. (37) by $\underline{\underline{\mathcal{U}}}_c^T \underline{\underline{\mathcal{J}}}$ and invoking the symplectic nature of matrix $\underline{\underline{\mathcal{U}}}_c$ leads to

$$\underline{\bar{\mathcal{F}}}_c^* = \underline{\underline{\bar{Z}}}^T \underline{\hat{P}}_c, \tag{42a}$$

$$\underline{\bar{\mathcal{U}}}_c^* = \underline{\underline{\bar{Y}}}^T \underline{\hat{u}}_c. \tag{42b}$$

Equation (42a) simply echoes eq. (28) and implies that generalized coordinates $\underline{\bar{\mathcal{F}}}_c^*$ are the stress resultants associated with the central solution. By construction, generalized coordinates $\underline{\bar{\mathcal{U}}}_c^*$, defined by eq. (42b), are energetically conjugated to the stress resultants and can be interpreted as average sectional rigid-body motions obtained as a linear combination of both nodal displacements and forces.

Consider now the following transformation of the system's Hamiltonian matrix

$$\underline{\underline{\bar{\mathcal{H}}}} \underline{\underline{\mathcal{U}}}_c = \underline{\underline{\mathcal{U}}}_c \begin{bmatrix} -\underline{\underline{\tilde{\bar{\mathcal{K}}}}}^* & \underline{\underline{\bar{S}}}^* \\ \underline{\underline{0}} & \underline{\underline{\tilde{\bar{\mathcal{K}}}}}^{*T} \end{bmatrix} = \underline{\underline{\mathcal{U}}}_c \underline{\underline{\bar{\mathcal{H}}}}_c, \tag{43}$$

The first six columns of matrix $\underline{\underline{\bar{\mathcal{H}}}}_c$ simply echo relationship (33). Because matrix $\underline{\underline{\mathcal{U}}}_c$ is symplectic, matrix $\underline{\underline{\bar{\mathcal{H}}}}_c$ must be Hamiltonian, see eq. (77), and its general structure is given by eq. (69). This implies that the lower-right partition of $\underline{\underline{\bar{\mathcal{H}}}}_c$ equals $\tilde{\bar{\mathcal{K}}}^{*T}$ and its upper-right partition, denoted $\underline{\underline{\bar{S}}}^*$, is symmetric. Matrix $\underline{\underline{\bar{S}}}^*$ will be interpreted later as the sectional compliance matrix.

Pre-multiplying eq. (43) by $\underline{\underline{\mathcal{U}}}_c^T \underline{\underline{\mathcal{J}}}$ and using eq. (24) yields

$$\underline{\underline{\bar{S}}}^* = \left[\underline{\underline{\bar{W}}}^T \quad \underline{\underline{\bar{Y}}}^T \right] \underline{\underline{\bar{\mathcal{F}}}} \begin{bmatrix} \underline{\underline{\bar{W}}} \\ \underline{\underline{\bar{Y}}} \end{bmatrix}. \tag{44}$$

Matrix $\underline{\underline{\bar{\mathcal{F}}}}$, defined by eq. (22), is symmetric and positive-definite; eq. (44) proves that the sectional compliance matrix, $\underline{\underline{\bar{S}}}^*$, shares the same properties, as expected. The existence of matrices $\underline{\underline{\bar{W}}}$, $\underline{\underline{\bar{Y}}}$, and $\underline{\underline{\bar{S}}}^*$ verifying eq. (43) and satisfying constraints (40a) and (40b) is proved in the next section.

4.3 Determination of the central solution

This section outlines the procedure for the determination of the central solution. Introducing the explicit expression of the Hamiltonian matrix, eq. (24), into system (43) yields two matrix equations,

$$\underline{\underline{\bar{Y}}} = \underline{\underline{\bar{C}}}^T \underline{\underline{\bar{W}}} + \underline{\underline{\bar{M}}}\, \underline{\underline{\bar{Z}}}\, \underline{\underline{\bar{S}}}^* + \underline{\underline{\bar{M}}}\, \underline{\underline{\bar{W}}}\, \underline{\underline{\tilde{\bar{\mathcal{K}}}}}^{*T}, \tag{45a}$$

$$\underline{\underline{\bar{E}}}\, \underline{\underline{\bar{W}}} = \underline{\underline{\bar{Y}}}\, \underline{\underline{\tilde{\bar{\mathcal{K}}}}}^{*T} - \underline{\underline{\bar{C}}}\, \underline{\underline{\bar{Z}}}\, \underline{\underline{\bar{S}}}^* - \underline{\underline{\bar{C}}}\, \underline{\underline{\bar{W}}}\, \underline{\underline{\tilde{\bar{\mathcal{K}}}}}^{*T}. \tag{45b}$$

First, the sectional compliance matrix is expressed in terms of $\underline{\underline{\bar{W}}}$ by pre-multiplying eq. (45a) by $\underline{\underline{\bar{Z}}}^T$ and using condition (40a) to find

$$\underline{\underline{\bar{S}}}^* = \underline{\underline{\bar{F}}}\left[\underline{\underline{I}} - (\underline{\underline{\bar{C}}}\,\underline{\underline{\bar{Z}}})^T\underline{\underline{\bar{W}}} - (\underline{\underline{\bar{M}}}\,\underline{\underline{\bar{Z}}})^T\underline{\underline{\bar{W}}}\,\tilde{\bar{\mathcal{K}}}^{*T}\right], \tag{46}$$

where the flexibility matrix is defined as

$$\underline{\underline{\bar{F}}} = (\underline{\underline{\bar{Z}}}^T\underline{\underline{\bar{M}}}\,\underline{\underline{\bar{Z}}})^{-1}. \tag{47}$$

Next, matrix $\underline{\underline{\bar{Y}}}$ is expressed in terms of $\underline{\underline{\bar{W}}}$ by introducing the sectional compliance matrix (46) into eqs. (45a) leading to eq. (48a). Finally, the governing equation for $\underline{\underline{\bar{W}}}$ is found by introducing eqs. (46) and (48a) into eq. (45b), leading to

$$\underline{\underline{\bar{Y}}} = \underline{\underline{\hat{C}}}^T\underline{\underline{\bar{W}}} + \underline{\underline{\hat{M}}}\,\underline{\underline{\bar{Z}}}\,\tilde{\bar{\mathcal{K}}}^{*T} + (\underline{\underline{\bar{M}}}\,\underline{\underline{\bar{Z}}})\underline{\underline{\bar{F}}}, \tag{48a}$$

$$\underline{\underline{\hat{M}}}\,\underline{\underline{\bar{W}}}\,\tilde{\bar{\mathcal{K}}}^{*T}\tilde{\bar{\mathcal{K}}}^{*T} - \underline{\underline{\hat{G}}}\,\underline{\underline{\bar{W}}}\,\tilde{\bar{\mathcal{K}}}^{*T} - \underline{\underline{\hat{E}}}\,\underline{\underline{\bar{W}}} = -\underline{\underline{\bar{V}}}, \tag{48b}$$

where the following notation was introduced

$$\underline{\underline{\hat{M}}} = \underline{\underline{\bar{M}}} - (\underline{\underline{\bar{M}}}\,\underline{\underline{\bar{Z}}})\underline{\underline{\bar{F}}}(\underline{\underline{\bar{M}}}\,\underline{\underline{\bar{Z}}})^T, \tag{49a}$$

$$\underline{\underline{\hat{C}}} = \underline{\underline{\bar{C}}} - (\underline{\underline{\bar{C}}}\,\underline{\underline{\bar{Z}}})\underline{\underline{\bar{F}}}(\underline{\underline{\bar{M}}}\,\underline{\underline{\bar{Z}}})^T, \tag{49b}$$

$$\underline{\underline{\hat{E}}} = \underline{\underline{\bar{E}}} - (\underline{\underline{\bar{C}}}\,\underline{\underline{\bar{Z}}})\underline{\underline{\bar{F}}}(\underline{\underline{\bar{C}}}\,\underline{\underline{\bar{Z}}})^T, \tag{49c}$$

$$\underline{\underline{\bar{V}}} = (\underline{\underline{\bar{M}}}\,\underline{\underline{\bar{Z}}})\underline{\underline{\bar{F}}}\tilde{\bar{\mathcal{K}}}^{*T} - (\underline{\underline{\bar{C}}}\,\underline{\underline{\bar{Z}}})\underline{\underline{\bar{F}}}, \tag{49d}$$

and $\underline{\underline{\hat{G}}} = \underline{\underline{\hat{C}}} - \underline{\underline{\hat{C}}}^T$. With the help of identities (30), it is shown easily that

$$\underline{\underline{\bar{Z}}}^T\underline{\underline{\hat{M}}} = \underline{\underline{\bar{Z}}}^T\underline{\underline{\hat{G}}} = \underline{\underline{\bar{Z}}}^T\underline{\underline{\hat{E}}} = \underline{\underline{0}}, \tag{50}$$

which proves that matrices $\underline{\underline{\hat{M}}}$, $\underline{\underline{\hat{G}}}$, and $\underline{\underline{\hat{E}}}$ are six times singular and matrix $\underline{\underline{\bar{Z}}}$ spans their null space. Furthermore, it is also true that

$$\underline{\underline{\bar{Z}}}^T\underline{\underline{\bar{V}}} = \underline{\underline{0}}. \tag{51}$$

System (48b) provides $6n$ equations for the $6n$ unknown components of matrix $\underline{\underline{\bar{W}}}$, but all matrices of the system are six times singular. For straight beams, $\tilde{\bar{\mathcal{K}}}^{*T}\tilde{\bar{\mathcal{K}}}^{*T} = \underline{\underline{0}}$, and system (48b) reduces to $\underline{\underline{\hat{G}}}\,\underline{\underline{\bar{W}}}\,\tilde{\bar{\mathcal{K}}}^{*T} + \underline{\underline{\hat{E}}}\,\underline{\underline{\bar{W}}} = \underline{\underline{\bar{V}}}$. The problem now splits into two subproblems, which can be solved recursively

$$\underline{\underline{\hat{E}}}\left[\underline{w}_4, \underline{w}_5, \underline{w}_6\right] = \left[\underline{v}_4, \underline{v}_5, \underline{v}_6\right], \tag{52a}$$

$$\underline{\underline{\hat{E}}}\left[\underline{w}_1, \underline{w}_2, \underline{w}_3\right] = \left[\underline{v}_1, \underline{v}_2, \underline{v}_3\right] + \underline{\underline{\hat{G}}}\left[\underline{w}_4, \underline{w}_5, \underline{w}_6\right]\tilde{t}, \tag{52b}$$

where $\underline{w}_i$ and $\underline{v}_i$ denote the i$^{\text{th}}$ column of matrices $\underline{\underline{\bar{W}}}$ and $\underline{\underline{\bar{V}}}$, respectively. Equations (50) show that the system matrix, $\underline{\underline{\hat{E}}}$, is six times singular. Linear systems (52a) and (52b) are, however, solvable because eqs (50) and (51) provide the solvability conditions.

Because the systems are singular, the general solutions can be written as $\underline{\underline{\bar{W}}} = \underline{\underline{\bar{W}}}^+ + \underline{\underline{\bar{Z}}}\,\underline{\underline{\alpha}}$, where $\underline{\underline{\bar{W}}}^+$ is a particular solution of the problem. Coefficients $\underline{\underline{\alpha}}$ are found easily as $\underline{\underline{\alpha}} = -\underline{\underline{\bar{Y}}}^T\underline{\underline{\bar{W}}}^+$ because the solution must satisfy condition (40b).

Equation (48a) now gives matrix $\underline{\underline{\bar{Y}}}$, which is independent of coefficients $\underline{\underline{\alpha}}$. Finally, eq. (46) yields the sectional compliance matrix as

$$\underline{\underline{\bar{S}}}^* = \underline{\underline{\bar{S}}}^+ - \tilde{\bar{\mathcal{K}}}^*\underline{\underline{\alpha}} - \underline{\underline{\alpha}}\,\tilde{\bar{\mathcal{K}}}^{*T}, \tag{53}$$

where $\underline{\underline{\bar{S}}}^+ = \underline{\underline{\bar{F}}}[\underline{\underline{I}} - (\underline{\underline{\bar{C}}}\,\underline{\underline{\bar{Z}}})^T\underline{\underline{\bar{W}}}^+ - (\underline{\underline{\bar{M}}}\,\underline{\underline{\bar{Z}}})^T\underline{\underline{\bar{W}}}^+\tilde{\bar{\mathcal{K}}}^{*T}]$. Equation (44) implies that the sectional compliance matrix obtained from eq. (53) is always symmetric and positive-definite.

4.4 Complete solution

The coordinate transformations defined in the two previous sections are now combined into a single transformation

$$\underline{\hat{\mathcal{X}}} = \underline{\underline{\mathcal{U}}}\,\underline{\mathcal{Q}} = \left[\underline{\underline{\mathcal{U}}}_c,\ \underline{\underline{\mathcal{U}}}_e\right]\underline{\mathcal{Q}}, \tag{54}$$

where array $\underline{\mathcal{Q}}^T = \{\underline{\mathcal{Q}}_c^T, \underline{\mathcal{Q}}_e^T\}$ stores the generalized coordinates associated with the central and extremity solutions. Because matrix $\underline{\underline{\mathcal{U}}}$ is symplectic, the following transformation results

$$\underline{\underline{\mathcal{U}}}^{-1}\underline{\underline{\bar{\mathcal{H}}}}\,\underline{\underline{\mathcal{U}}} = \underline{\underline{\hat{\mathcal{H}}}} = \begin{bmatrix} \underline{\underline{\bar{\mathcal{H}}}}_c & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\bar{\mathcal{H}}}}_e \end{bmatrix}. \tag{55}$$

Note that the reduced system Hamiltonian is now in nearly-diagonal form, enabling an analytical solution of the problem.

5 Closed-form solutions

The reduced governing equations of the problem given by eq. (55) are now fully defined and closed-form solutions are derived in this section. Because the central and extremity solutions are decoupled, their solutions can be derived independently.

5.1 Extremity solutions

Because the portion of the Hamiltonian pertaining to the extremity solution is diagonal, see eq. (36), the governing equations for the generalized coordinates are of the form $q'_{+\lambda_i} = \bar{\lambda}_i q_{+\lambda_i}$ and their solutions are $q_{+\lambda_i} = \ell\exp(+\bar{\lambda}_i\bar{\alpha}_1)$. The corresponding closed-form solution in the physical domain is then obtained easily

$$\underline{\hat{\mathcal{X}}}_e = \sum_{i=1}^{N_e}\left[e^{+\bar{\lambda}_i\bar{\alpha}_1}\underline{\mathcal{U}}_{+\lambda_i}\ell_{+\lambda_i} + e^{-\bar{\lambda}_i\bar{\alpha}_1}\underline{\mathcal{U}}_{-\lambda_i}\ell_{-\lambda_i}\right], \tag{56}$$

where $N_e = n - 6$ is the number of extremity solutions, and $\ell_{+\lambda_i}$ and $\ell_{-\lambda_i}$ are integration constants to be evaluated from the boundary conditions. Clearly, these solutions are exponentially decaying solutions emanating from the beam's ends.

5.2 Central solutions

The twelve governing differential equations for the central solutions are $\underline{\mathcal{Q}}'_c = \underline{\underline{\bar{\mathcal{H}}}}_c\underline{\mathcal{Q}}_c$. The last six equations of this set provide the beam's equilibrium equations

$$\underline{\bar{\mathcal{F}}}_c^{*\prime} - \underline{\underline{\tilde{\bar{\mathcal{K}}}}}^{*T}\underline{\bar{\mathcal{F}}}_c^{*} = \underline{0}, \tag{57}$$

which can also be obtained from elementary equilibrium considerations directly. The first six equations define the sectional constitutive laws relating the beam's stress resultants, $\underline{\bar{\mathcal{F}}}_c^*$, to the sectional strains, $\underline{\mathcal{E}}_c^*$, as $\underline{\mathcal{E}}_c^* = \underline{\underline{\bar{S}}}^*\underline{\bar{\mathcal{F}}}^* = \underline{\bar{\mathcal{U}}}_c^{*\prime} + \tilde{\bar{\mathcal{K}}}^*\underline{\bar{\mathcal{U}}}_c^*$. The sectional strains are defined as

$$\underline{\mathcal{E}}_c^* = \underline{\bar{\mathcal{U}}}_c^{*\prime} + \tilde{\bar{\mathcal{K}}}^*\underline{\bar{\mathcal{U}}}_c^*, \tag{58}$$

and the sectional constitutive laws become

$$\underline{\mathcal{E}}_c^* = \underline{\underline{\bar{S}}}^*\underline{\bar{\mathcal{F}}}_c^*. \tag{59}$$

Clearly, symmetric matrix $\underline{\underline{\bar{S}}}^*$ stores the components of the sectional compliance matrix resolved in the material basis.

The general form of the central solutions is $\underline{\mathcal{Q}}_c = \exp(\bar{\alpha}_1 \underline{\underline{\bar{\mathcal{H}}}}_c)\underline{\mathcal{Q}}_{c0}$, where array $\underline{\mathcal{Q}}_{c0}$ stores arbitrary integration constants and can be interpreted as the value of the generalized coordinates at $\bar{\alpha}_1 = 0$, *i.e.*, $\underline{\mathcal{Q}}_{c0}^T = \{\underline{\bar{\mathcal{U}}}_0^{*T}, \underline{\bar{\mathcal{F}}}_0^*\}$. The central solutions then become

$$\underline{\hat{\mathcal{X}}}_c(\bar{\alpha}_1) = \underline{\underline{\mathcal{U}}}_c \exp(\bar{\alpha}_1 \underline{\underline{\bar{\mathcal{H}}}}_c)\underline{\mathcal{Q}}_{c0}. \tag{60}$$

To obtain a closed-form solution of the problem, the exponential matrix must be evaluated. The specific structure of matrix $\underline{\underline{\bar{\mathcal{H}}}}_c$ defined by eq. (43) implies the following result

$$\exp(\bar{\alpha}_1 \underline{\underline{\bar{\mathcal{H}}}}_c) = \begin{bmatrix} \underline{\underline{\mathcal{C}}}^{-1} & \underline{\underline{\mathcal{B}}} \\ \underline{\underline{0}} & \underline{\underline{\mathcal{C}}}^T \end{bmatrix}, \tag{61}$$

where the motion tensor [26], denoted $\underline{\underline{\mathcal{C}}}$, is defined as $\underline{\underline{\mathcal{C}}}(\bar{\alpha}_1) = \exp(\bar{\alpha}_1 \underline{\underline{\tilde{\bar{\mathcal{K}}}}}^*)$. Closed-form solutions for the motion tensor and for matrix $\underline{\underline{\mathcal{B}}}$ are given in appendix C and D, respectively. With this result, eq. (60) expands to

$$\underline{\hat{u}}_c = \underline{\underline{\bar{Z}}}\,\underline{\underline{\mathcal{C}}}^{-1}\underline{\bar{\mathcal{U}}}_0^* + (\underline{\underline{\bar{Z}}}\,\underline{\underline{\mathcal{B}}} + \underline{\underline{\bar{W}}}\,\underline{\underline{\mathcal{C}}}^T)\underline{\bar{\mathcal{F}}}_0^*, \tag{62a}$$

$$\underline{\hat{P}}_c = \underline{\underline{\bar{Y}}}\,\underline{\underline{\mathcal{C}}}^T\underline{\bar{\mathcal{F}}}_0^*, \tag{62b}$$

where $\underline{\bar{\mathcal{U}}}_0^* = \underline{\bar{\mathcal{U}}}_c^*(\bar{\alpha}_1 = 0)$ and $\underline{\bar{\mathcal{F}}}_0^* = \underline{\bar{\mathcal{F}}}_c^*(\bar{\alpha}_1 = 0)$.

The closed-form of the central solution can now be derived. First, the internal forces in the beam are found by solving eq. (57) to find

$$\underline{\bar{\mathcal{F}}}_c^*(\bar{\alpha}_1) = \underline{\underline{\mathcal{C}}}^T(\bar{\alpha}_1)\underline{\bar{\mathcal{F}}}_0^*, \tag{63}$$

where the motion tensor, $\underline{\underline{\mathcal{C}}}(\bar{\alpha}_1)$, is given by eq. (79). Next, the rigid-body motions of the section are obtained by introducing eqs. (62) into eq. (42b) to find

$$\underline{\bar{\mathcal{U}}}_c^*(\bar{\alpha}_1) = \underline{\underline{\mathcal{C}}}^{-1}\underline{\bar{\mathcal{U}}}_0^* + \underline{\underline{\mathcal{B}}}\,\underline{\bar{\mathcal{F}}}_0^*. \tag{64}$$

The first term represents the effects of the boundary conditions at $\bar{\alpha}_1 = 0$ and the second the contributions of the elastic deformation of the beam. Without loss of generality, it can be assumed that the rigid-body motions are constrained at $\bar{\alpha}_1 = 0$, *i.e.*, $\underline{\bar{\mathcal{U}}}_0^* = \underline{0}$, and eq. (64) then reduces to $\underline{\bar{\mathcal{U}}}^*(\bar{\alpha}_1) = \underline{\underline{\mathcal{B}}}(\bar{\alpha}_1)\underline{\bar{\mathcal{F}}}_0^*$. Clearly, matrix $\underline{\underline{\mathcal{B}}}(\bar{\alpha}_1)$ can be interpreted as a transfer matrix that evaluates the distribution of rigid-body motions along the beam's span given units sectional forces at $\bar{\alpha}_1 = 0$.

The nodal force distribution over the beam's cross-section is obtained from eq. (62b) as

$$\underline{\hat{P}}_c(\bar{\alpha}_1) = \underline{\underline{\bar{Y}}}\,\underline{\bar{\mathcal{F}}}_c^*(\bar{\alpha}_1). \tag{65}$$

Matrix $\underline{\underline{\bar{Y}}}$ describes the distribution of nodal forces associated with unit sectional stress resultants. Finally, the nodal displacement distribution over the beam's cross-section is obtained from eq. (62a) as

$$\underline{\hat{u}}_c(\bar{\alpha}_1) = \underline{\underline{\bar{Z}}}\,\underline{\bar{\mathcal{U}}}_c^*(\bar{\alpha}_1) + \underline{\underline{\bar{W}}}\,\underline{\bar{\mathcal{F}}}_c^*(\bar{\alpha}_1). \tag{66}$$

The first term represents the contribution from the rigid-body motions and matrix $\underline{\underline{\bar{W}}}$ describes the cross-sectional warping associated with unit sectional stress resultants. Of course, the three-dimensional strain distribution in the beam can be recovered by introducing the nodal displacements, $\underline{\hat{u}}_c(\bar{\alpha}_1)$, into eq. (14).

6 Numerical results

The first numerical example deals with the torsion of a bar presenting a rectangular cross-section of width a and height b and made of a homogenous, isotropic material of shear modulus G. The solution of the problem is given by Saint-Venant's theory for torsion, and in this case, the torsional stiffness of the section can be expressed as a series expansion [1]. The proposed approach was used to determine the torsional stiffness of the structure; the rectangular cross-section was discretized using nine-noded quadrilateral elements. The following meshes, 40×40, 40×60, 40×80, 40×120, 40×140, 40×160, 40×200, were used for aspect ratios $a/b = 1$, 3, 5, 7, 9, and 11, respectively. Figure 3 shows the non-dimensional sectional torsion stiffness, $H_{11}/(Gab^3)$, versus the aspect ratio, a/b.

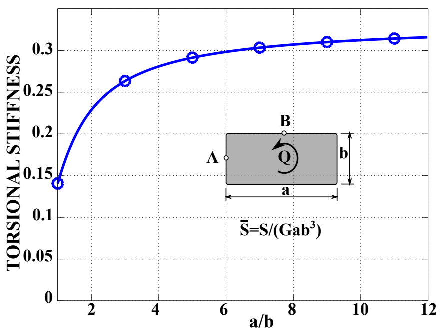

Figure 3: Non-dimensional torsional stiffness versus aspect ratio. Analytical solution: solid line; present solution: symbols ◦.

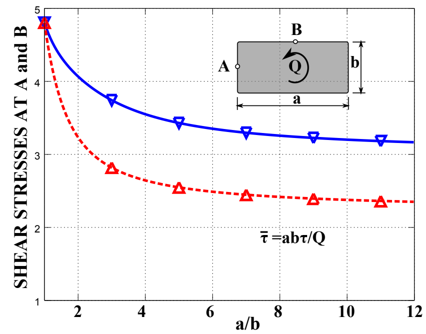

Figure 4: Shear stresses at points **A** and **B**. Analytical solution: solid line (point **B**), dashed line (point **A**); present solution: symbols ▽ (point **B**), △ (point **A**).

Next, the non-dimensional shear stresses, $ab^2\tau/Q$, at the mid-points of the shortest and longest edges of the section, denoted points **A** and **B**, respectively, were evaluated for an applied torque, Q. The predictions of the present approach were compared with those obtained from the series solution in fig. 4, which shows the non-dimensional shear stresses at points **A** and **B** for the two approaches. The present solution is in good agreement with the analytical solution. Note that both Saint-Venant's theory for torsion and the present central solution are functions of the loading only, in this case, the applied torque. Both solutions are valid "far away from the beam's ends" and hence, are not subjected to boundary conditions. Only the extremity solutions are affected by the boundary conditions.

Next, a beam with an I-shaped cross-section is investigated. Figure 5 shows the configuration of the section, where the height h and width b both equal 200 mm and the thickness $t = 5$ mm. The web, top and bottom flanges consist of 8 plies of Graphite/Epoxy material. Each ply has the same thickness, $t_p = 0.625$ mm. Graphite/Epoxy has the following material stiffness properties: longitudinal, transverse, and shear moduli are $E_L = 181$, $E_T = 10.3$, and $G_{LT} = 7.17$ GPa, respectively; Poisson's ratios are $\nu_{LT} = 0.28$ and $\nu_{TN} = 0.33$. The lay-ups defined in fig. 5 start with the innermost ply and end with the outermost ply; 0° fibers are aligned with the axis of the beam and a positive ply angle indicates a right-hand rotation about the local normal to the thin wall. Nine-node quadrilateral elements arranged in a 80×16 grid are used to mesh the web, and top and bottom flanges.

The diagonal entries of the predicted stiffness matrix are $C^*_{11} = 242.3$, $C^*_{22} = 42.99$, $C^*_{33} = 22.94$ MN, $C^*_{44} = 0.4733$, $C^*_{55} = 1899$, and $C^*_{66} = 586.2$ kN·m^2. In addition, three off-diagonal terms did not vanish: $C^*_{14} = 5.392$, $C^*_{25} = $ -9.341, and $C^*_{36} = 3.058$ kN·m. Figure 6 depicts the warping field of the cross section under a unit shear force along unit vector $\bar{\imath}_3$. As shown in the figure, complex, three-dimensional deformation arises near the connections of the web and flanges.

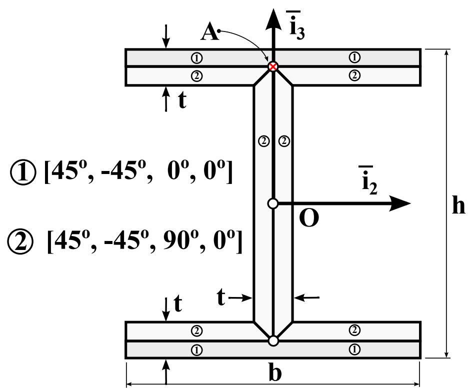

Figure 5: Configuration of the I-section beam.

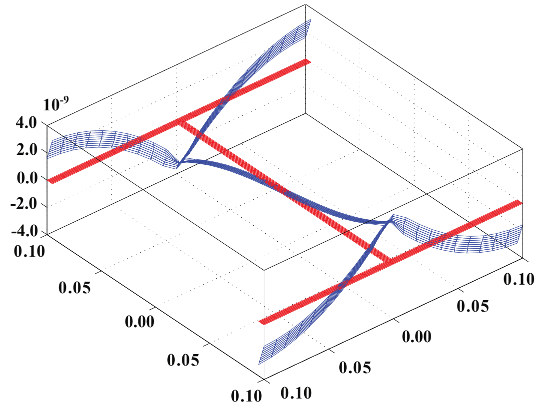

Figure 6: Warping of the I-section beam.

At point **A**, of coordinates $x_2 = 0$ and $x_3 = 97.5$ mm, indicated on the figure, the out-of-plane stress components generated by an applied torque of unit magnitude are $\sigma_{11} = -26.51$, $\tau_{12} = -18.07$, and $\tau_{13} = -1.586$ kPa. Elastic couplings give rise to an axial stress component, σ_{11}, whose magnitude is larger than those of the shear stress components, τ_{12} and τ_{13}, resulting from the applied torque. The in-plane stress components are $\sigma_{22} = -3.239$, $\tau_{23} = -0.5937$, and $\sigma_{33} = -3.763$ kPa. In classical beam theories, these in-plane stress components are assumed to vanish; yet, the present analysis shows stress components are not negligible. This clearly demonstrates the shortcomings of the assumptions on which classical beam theories are based. When dealing with complex cross-sections made of anisotropic materials, these assumptions are no longer justified and the accuracy of the stress predictions is no longer guaranteed.

7 Discussion

From the onset, the proposed approach is based on a Hamiltonian formulation of the problem and progresses to the solution through a set of structure preserving transformations. The null and non-vanishing eigenvalues of the system Hamiltonian matrix give rise to central and extremity solutions, respectively, which are characterized mathematically by polynomial and exponential functions, respectively. These two types of solutions are a natural consequence of Saint-Venant's principle [7].

The central solutions are exact solutions of the linear theory of three-dimensional elasticity for beams presenting uniform geometric and material characteristics along their span and are valid far away from the beam's edges, where all extremity solutions become negligible. Analytical expressions for the spanwise variation of the central solutions are derived in this paper and their distributions over the cross-section are obtained from a finite element model. The accuracy of the solutions is limited by the discretization inherent to the finite element method only.

The symplectic coordinate transformation that characterizes the present approach preserves the Hamiltonian nature of the problem and its associated physical characteristics. This approach is rooted in the work of Zhong *et al.* [11, 12, 13], but bypasses the need to determine the Jordan canonical form of the Hamiltonian. This task is known to be computationally unstable and hence, Morandini *et al.* [14] had to rely on advanced numerical tools to evaluate this Jordan form.

The present approach is markedly different from the variational asymptotic method developed by Hodges and co-workers [19], which is based on an expansion of the solution to a certain order of one or more small parameters. The asymptotic approach is inherently approximate as it represents a truncation of the exact solution to a certain level of accuracy.

An interesting manifestation of the difference between the two approaches is the treatment of shear deformations. In the present approach, shear deformations are part of the central solution because their spanwise variation is polynomial: transverse deflections due to shear do not decay exponentially away from the beam's edges. This does not imply, however, that these deflections are large. Indeed, it is well known that shear deflections are far smaller that those due to bending [1]. In contrast, the variational asymptotic method expands the solution in terms of a small parameter, $\varepsilon = a_r/L$, and correctly concludes that to the first order, shear effects are negligible. A refined theory that includes such effect within the framework of asymptotic methods is given by Popescu and Hodges [27].

Because the central solutions are exact solutions of three-dimensional elasticity for long, uniform beams undergoing small strain deformations, it is reasonable to require beam theories used for flexible multibody and structural dynamics simulations to reproduce these solutions under the same conditions. The analytical solutions for the generalized coordinates given by eqs. (63) and (64) are polynomial in nature and can be reproduced exactly by classical finite element models based on polynomial shape functions, provided that the sectional compliance matrix defined by eq. (53) is used in the computation. The detailed three-dimensional force and displacement fields can then be recovered from eqs. (65) and (66), respectively. These tasks are conveniently performed as post-processing steps to the beam analysis.

Note that the central solutions expressed by eqs. (63) to (66) are completely defined by the six sectional stress resultants, $\underline{\bar{\mathcal{F}}}^*_c(\bar{\alpha}_1)$, or alternatively, by the six sectional strains, $\underline{\mathcal{E}}^*_c$, see eq. (59). Consequently, six parameters are sufficient to describe the kinematics of the problem. A popular choice is to select three displacements and three rotations, as is commonly done for geometrically exact beams [3, 28]. Redundant sets can also be used if an adequate number of kinematic constraint are imposed, see Betsch and Steinmann [29], as long as the sectional strains can be evaluated unequivocally. If additional strain measures are defined, they should be used to describe the behavior of extremity solutions, as discussed by Yu *et al.* [30], who generalized Vlassov beam theory for composite structures.

Many of the approaches presently used in flexible multibody and structural dynamics simulations are based on inconsistent assumptions. Often, to obtain simple kinematic representations, the beam's cross-section is assumed to remain plane and its displacement field is fully defined by three displacement and three rotation components. On the other hand, its sectional stiffnesses are evaluated based on mechanics of material theories that rely on more sophisticated representation of the deformation. Such hybrid approaches can lead to significant inaccuracies, specially for thin-walled beams, or beams made of anisotropic materials.

Three-dimensional warping fields, such as those depicted in figs 6, are indispensable for recovering accurate three-dimensional stress fields. Although these warping fields are complex, they depend on the geometric and material characteristics of the beam's cross-section only. The simplified kinematics used as the basis for many beam formulations should not be interpreted as the cross-sectional rigid-body motions but rather as the rigid-body motions energetically conjugated the the sectional resultants, see eq. (41); they can be obtained from the actual displacement field by means of eq. (42b).

The proposed approach can be interpreted as a multi-scale procedure. The "local-scale" problem is the two-dimensional finite element analysis of the cross-section and the "global-scale" problem is the one-dimensional beam analysis. For the linear problem described here, no iteration is required between the local- and global-scale problems. The local-scale problem provides the sectional stiffness matrix required to carry out the global-scale analysis. Because the energy equivalence condition (41) is an inherent part of the solution, the strain energy associated with the central solution at the local scale is identical to that evaluated at the global scale. Finally, given the sectional stress resultants evaluated at the global scale, the local-scale nodal forces and displacements are recovered in a post-processing operation by eqs. (65) and (66), respectively.

8 Conclusions

This paper has presented a novel method for the analysis of beams based on the Hamiltonian formalism. The approach proceeds through a set of structure preserving transformations using symplectic matrices and decomposes the solution into its central and extremity components. The central solution is an exact solution of the linear theory of three-dimensional elasticity for beams presenting uniform geometric and material characteristics along their span and is valid far away from the beam's edges, where all extremity solutions become negligible.

The kinematic assumptions underpinning commonly used beam theories have been eliminated altogether and yet, exact solutions are obtained for the central behavior of the beam. Analytical expressions for the spanwise variation of the central solutions are derived and their distributions over the cross-section are obtained from a two-dimensional finite element model. The accuracy of the solution is limited by the discretization inherent to the finite element method only.

Numerical examples have been presented to demonstrate the capabilities of the analysis. Predictions were compared to exact solutions of elasticity. Because it requires a two-dimensional discretization only, the proposed approach is very efficient and requires far less computational effort than approaches based on three-dimensional finite elements.

The proposed approach can be generalized to naturally curved and twisted beams undergoing large displacements and rotations but small strains, leading to a geometrically exact formulation that can be used for flexible multibody dynamics applications.

A Hamiltonian matrices

Matrix $\underline{\underline{\bar{\mathcal{H}}}}$, of size $2n \times 2n$, is said to be Hamiltonian if it satisfies the following property

$$(\underline{\underline{\mathcal{J}}}\,\underline{\underline{\bar{\mathcal{H}}}})^T = \underline{\underline{\mathcal{J}}}\,\underline{\underline{\bar{\mathcal{H}}}}, \tag{67}$$

where skew-symmetric matrix $\underline{\underline{\mathcal{J}}}$ is defined as

$$\underline{\underline{\mathcal{J}}} = \begin{bmatrix} \underline{\underline{0}}_{n\times n} & \underline{\underline{I}}_{n\times n} \\ -\underline{\underline{I}}_{n\times n} & \underline{\underline{0}}_{n\times n} \end{bmatrix}. \tag{68}$$

Note the following properties of matrix $\underline{\underline{\mathcal{J}}}$, $\underline{\underline{\mathcal{J}}}^T = -\underline{\underline{\mathcal{J}}}$ and $\underline{\underline{\mathcal{J}}}\,\underline{\underline{\mathcal{J}}}^T = \underline{\underline{\mathcal{I}}}$. In view of these properties, the definition (67) of Hamiltonian matrices can be recast as $\underline{\underline{\bar{\mathcal{H}}}} = \underline{\underline{\mathcal{J}}}\,\underline{\underline{\bar{\mathcal{H}}}}^T\underline{\underline{\mathcal{J}}}$. Because matrix $\underline{\underline{\mathcal{J}}}$ is skew-symmetric, $\underline{\mathcal{U}}^T\underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}} = 0$ for any vector $\underline{\mathcal{U}}$.

Property (67) implies that the most general form of a Hamiltonian matrix is

$$\underline{\underline{\bar{\mathcal{H}}}} = \begin{bmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{C}} & -\underline{\underline{A}}^T \end{bmatrix}, \tag{69}$$

where matrices $\underline{\underline{A}}$, $\underline{\underline{B}}$, and $\underline{\underline{C}}$ are of size $n \times n$, and matrices $\underline{\underline{B}}$ and $\underline{\underline{C}}$ are symmetric. Clearly, the transpose of a Hamiltonian matrix is also Hamiltonian.

A.1 Eigenvalues of Hamiltonian matrices

Let λ and μ be two eigenvalues of a Hamiltonian matrix and the associated eigenvectors are denoted $\underline{\mathcal{U}}_\lambda$ and $\underline{\mathcal{U}}_\mu$, respectively, *i.e.*, $\underline{\underline{\bar{\mathcal{H}}}}\,\underline{\mathcal{U}}_\lambda = \lambda\underline{\mathcal{U}}_\lambda$ and $\underline{\underline{\bar{\mathcal{H}}}}\,\underline{\mathcal{U}}_\mu = \mu\underline{\mathcal{U}}_\mu$. Pre-multiplying the first equation by $\underline{\mathcal{U}}_\mu^T\underline{\underline{\mathcal{J}}}$ and the second by $\underline{\mathcal{U}}_\lambda^T\underline{\underline{\mathcal{J}}}$ leads to $\underline{\mathcal{U}}_\mu^T\underline{\underline{\mathcal{J}}}\,\underline{\underline{\bar{\mathcal{H}}}}\,\underline{\mathcal{U}}_\lambda = \lambda\underline{\mathcal{U}}_\mu^T\underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_\lambda$ and $\underline{\mathcal{U}}_\lambda^T\underline{\underline{\mathcal{J}}}\,\underline{\underline{\bar{\mathcal{H}}}}\,\underline{\mathcal{U}}_\mu = \mu\underline{\mathcal{U}}_\lambda^T\underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_\mu$, respectively. Because matrix $\underline{\underline{\mathcal{J}}}\,\underline{\underline{\bar{\mathcal{H}}}}$ is symmetric, the right-hand sides of these two equations are identical and subtraction yields $(\lambda+\mu)\underline{\mathcal{U}}_\mu^T \underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_\lambda = 0$. If $\lambda+\mu \neq 0$, the following symplectic orthogonality results

$$\underline{\mathcal{U}}_\mu^T \underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_\lambda = 0. \tag{70}$$

Let λ be an eigenvalue of a Hamiltonian matrix associated with eigenvector $\underline{\mathcal{U}}_{+\lambda}$, *i.e.*, $\underline{\underline{\bar{\mathcal{H}}}}\,\underline{\mathcal{U}}_{+\lambda} = \lambda\underline{\mathcal{U}}_{+\lambda}$. It then follows that $\underline{\underline{\mathcal{J}}}\,\underline{\underline{\bar{\mathcal{H}}}}\,\underline{\mathcal{U}}_{+\lambda} = \lambda\underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_{+\lambda}$ and property (67) then implies $\underline{\underline{\bar{\mathcal{H}}}}^T(\underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_{+\lambda}) = -\lambda(\underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_{+\lambda})$. Clearly, if $\underline{\mathcal{U}}_{+\lambda}$ is an eigenvector of $\underline{\underline{\bar{\mathcal{H}}}}$ associated with eigenvalue λ, $\underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_{+\lambda}$ is an eigenvector of $\underline{\underline{\bar{\mathcal{H}}}}^T$ associated with eigenvalue $-\lambda$. Because the spectra of eigenvalues of matrices $\underline{\underline{\bar{\mathcal{H}}}}$ and $\underline{\underline{\bar{\mathcal{H}}}}^T$ are identical, the eigenvalues of Hamiltonian matrices are symmetric about the imaginary axis, *i.e.*, occur in pairs of opposite sign, $\pm\lambda$. In summary,

$$\left(\underline{\underline{\bar{\mathcal{H}}}} - \lambda\underline{\underline{\mathcal{I}}}\right)\underline{\mathcal{U}}_{+\lambda} = \underline{0} \Longleftrightarrow \left(\underline{\underline{\bar{\mathcal{H}}}}^T + \lambda\underline{\underline{I}}\right)(\underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_{+\lambda}) = \underline{0}, \tag{71a}$$

$$\left(\underline{\underline{\bar{\mathcal{H}}}} + \lambda\underline{\underline{\mathcal{I}}}\right)\underline{\mathcal{U}}_{-\lambda} = \underline{0} \Longleftrightarrow \left(\underline{\underline{\bar{\mathcal{H}}}}^T - \lambda\underline{\underline{I}}\right)(\underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_{-\lambda}) = \underline{0}, \tag{71b}$$

where eqs (71a) and (71b) express identical properties for eigenvalues $+\lambda$ and $-\lambda$, respectively. Vectors $\underline{\mathcal{U}}_{+\lambda}$ and $\underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_{-\lambda}$ can also be interpreted as the right and left eigenvectors of matrix $\underline{\underline{\bar{\mathcal{H}}}}$ both associated with eigenvalue $+\lambda$.

It is convenient to define matrix $\underline{\underline{\mathcal{U}}}_{\mp\lambda} = \left[\underline{\mathcal{U}}_{-\lambda}, \underline{\mathcal{U}}_{+\lambda}\right]$, whose columns store the right eigenvectors associated with eigenvalues $-\lambda$ and $+\lambda$. These vectors enjoy the following properties

$$\underline{\underline{\mathcal{U}}}_{\mp\lambda}^T \underline{\underline{\mathcal{J}}}\,\underline{\underline{\mathcal{U}}}_{\mp\lambda} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{72}$$

Indeed, $\underline{\mathcal{U}}_{+\lambda}^T \underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_{+\lambda} = \underline{\mathcal{U}}_{-\lambda}^T \underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_{-\lambda} = 0$ and vectors $\underline{\mathcal{U}}_{+\lambda}$ and $\underline{\mathcal{U}}_{-\lambda}$ can be normalized to enforce $\underline{\mathcal{U}}_{-\lambda}^T \underline{\underline{\mathcal{J}}}\,\underline{\mathcal{U}}_{+\lambda} = 1$. The following pseudo-orthogonality statement in the space of Hamiltonian matrix then follows

$$\underline{\underline{\mathcal{U}}}_{\mp\lambda}^T \underline{\underline{\mathcal{J}}}\,\underline{\underline{\bar{\mathcal{H}}}}\,\underline{\underline{\mathcal{U}}}_{\mp\lambda} = \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}. \tag{73}$$

B Symplectic matrices

Matrix $\underline{\underline{\mathcal{S}}}$, of size $2n \times 2n$, is said to be symplectic if it satisfies the following property

$$\underline{\underline{\mathcal{S}}}^T \underline{\underline{\mathcal{J}}}\,\underline{\underline{\mathcal{S}}} = \underline{\underline{\mathcal{J}}}, \tag{74}$$

where skew-symmetric matrix $\underline{\underline{\mathcal{J}}}$ is defined by eq. (68). The inverse of a symplectic matrix is expressed easily as

$$\underline{\underline{\mathcal{S}}}^{-1} = \underline{\underline{\mathcal{J}}}^T \underline{\underline{\mathcal{S}}}^T \underline{\underline{\mathcal{J}}}. \tag{75}$$

Indeed, using definition (74) leads to $\underline{\underline{\mathcal{S}}}^{-1}\underline{\underline{\mathcal{S}}} = \underline{\underline{\mathcal{J}}}^T \underline{\underline{\mathcal{S}}}^T \underline{\underline{\mathcal{J}}}\,\underline{\underline{\mathcal{S}}} = \underline{\underline{\mathcal{J}}}^T \underline{\underline{\mathcal{J}}} = \underline{\underline{\mathcal{I}}}$.

Recasting eq. (74) as $\underline{\underline{\mathcal{J}}} = \underline{\underline{\mathcal{S}}}^{-T} \underline{\underline{\mathcal{J}}}\,\underline{\underline{\mathcal{S}}}^{-1}$ and using eq. (75) to express the inverse leads to $\underline{\underline{\mathcal{J}}} = (\underline{\underline{\mathcal{J}}}^T \underline{\underline{\mathcal{S}}}^T \underline{\underline{\mathcal{J}}})^T \underline{\underline{\mathcal{J}}}\,(\underline{\underline{\mathcal{J}}}^T \underline{\underline{\mathcal{S}}}^T \underline{\underline{\mathcal{J}}})$, and finally

$$\underline{\underline{\mathcal{S}}}\,\underline{\underline{\mathcal{J}}}\,\underline{\underline{\mathcal{S}}}^T = \underline{\underline{\mathcal{J}}}. \tag{76}$$

This means that the transpose of a symplectic matrix is itself symplectic. It is proved easily that if matrix $\underline{\underline{\mathcal{S}}}$ is symplectic, its inverse is also symplectic. Furthermore, the product of two symplectic matrices is also symplectic. Note that matrices $\underline{\underline{\mathcal{J}}}$ and $\underline{\underline{\mathcal{I}}}$ are symplectic.

A close connection exists between symplectic and Hamiltonian matrices. First, their definitions are closely related: $\underline{\underline{\bar{\mathcal{H}}}} = \underline{\underline{\mathcal{J}}}\,\underline{\underline{\bar{\mathcal{H}}}}^T \underline{\underline{\mathcal{J}}}$ and $\underline{\underline{\mathcal{S}}}^T \underline{\underline{\mathcal{J}}}\,\underline{\underline{\mathcal{S}}} = \underline{\underline{\mathcal{J}}}$, for Hamiltonian and symplectic matrices, respectively. Next, consider matrix $\hat{\underline{\underline{\mathcal{H}}}}$, defined through the following transformation, $\hat{\underline{\underline{\mathcal{H}}}} = \underline{\underline{\mathcal{S}}}^{-1}\bar{\underline{\underline{\mathcal{H}}}}\,\underline{\underline{\mathcal{S}}}$. Using eq. (75) leads to $\underline{\underline{\mathcal{J}}}\,\hat{\underline{\underline{\mathcal{H}}}} = \underline{\underline{\mathcal{J}}}(\underline{\underline{\mathcal{J}}}^T\underline{\underline{\mathcal{S}}}^T\underline{\underline{\mathcal{J}}})\bar{\underline{\underline{\mathcal{H}}}}\,\underline{\underline{\mathcal{S}}} = \underline{\underline{\mathcal{S}}}^T(\underline{\underline{\mathcal{J}}}\,\bar{\underline{\underline{\mathcal{H}}}})\underline{\underline{\mathcal{S}}}$, which implies that matrix $\underline{\underline{\mathcal{J}}}\,\hat{\underline{\underline{\mathcal{H}}}}$ is symmetric if matrix $(\underline{\underline{\mathcal{J}}}\,\bar{\underline{\underline{\mathcal{H}}}})$ is itself symmetric. Consequently, if matrix $\bar{\underline{\underline{\mathcal{H}}}}$ is Hamiltonian, so is matrix $\hat{\underline{\underline{\mathcal{H}}}}$. It follows that the transformation of a Hamiltonian matrix by a symplectic matrix yields a Hamiltonian matrix, *i.e.*,

$$\hat{\underline{\underline{\mathcal{H}}}} = \underline{\underline{\mathcal{S}}}^{-1}\bar{\underline{\underline{\mathcal{H}}}}\,\underline{\underline{\mathcal{S}}}. \tag{77}$$

Consider the following partition of a symplectic matrix

$$\underline{\underline{\mathcal{S}}} = \begin{bmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{C}} & \underline{\underline{D}} \end{bmatrix}, \tag{78}$$

where matrices $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$, and $\underline{\underline{D}}$ are of size $n \times n$. The definition of a symplectic matrix then implies $\underline{\underline{A}}^T\underline{\underline{D}} - \underline{\underline{C}}^T\underline{\underline{B}} = \underline{\underline{I}}$, $(\underline{\underline{A}}^T\underline{\underline{C}}) = (\underline{\underline{A}}^T\underline{\underline{C}})^T$, $(\underline{\underline{B}}^T\underline{\underline{D}}) = (\underline{\underline{B}}^T\underline{\underline{D}})^T$. The last two properties imply that products $\underline{\underline{A}}^T\underline{\underline{C}}$ and $\underline{\underline{B}}^T\underline{\underline{D}}$ form symmetric matrices.

C Exponential of the generalized skew-symmetric matrix

Next, consider an ordinary differential equation with constant coefficients of the form $\underline{\mathcal{U}}' = \tilde{\bar{\underline{\underline{\mathcal{K}}}}}^*\underline{\mathcal{U}}$, where $\tilde{\bar{\underline{\underline{\mathcal{K}}}}}^*$ is the generalized skew-symmetric matrix defined by eq. (26). The general solution of this equation is $\underline{\mathcal{U}}(x) = \exp(\tilde{\bar{\underline{\underline{\mathcal{K}}}}}^* x)\underline{\mathcal{U}}_0$, where array $\underline{\mathcal{U}}_0$ stores the integration constants. It is shown easily that $\tilde{\bar{\underline{\underline{\mathcal{K}}}}}^{*2} = \underline{\underline{0}}$, and the expression for the exponential reduces to

$$\underline{\underline{\mathcal{C}}}(x) = \exp(\tilde{\bar{\underline{\underline{\mathcal{K}}}}}^* x) = \underline{\underline{I}} + x\tilde{\bar{\underline{\underline{\mathcal{K}}}}}^*, \tag{79}$$

where $\underline{\underline{\mathcal{C}}}$ is the motion tensor [26].

D Exponential of the Hamiltonian matrix

Finally, consider an ordinary differential equation with constant coefficients of the form $\underline{\mathcal{Q}}' = \bar{\underline{\underline{\mathcal{H}}}}_c\underline{\mathcal{Q}}$, where $\bar{\underline{\underline{\mathcal{H}}}}_c$ is the Hamiltonian matrix defined by eq. (43). The general solution of this equation is $\underline{\mathcal{Q}}(x) = \exp(\bar{\underline{\underline{\mathcal{H}}}}_c x)\underline{\mathcal{Q}}_0$, where array $\underline{\mathcal{Q}}_0$ stores the integration constants. It is shown easily that $\bar{\underline{\underline{\mathcal{H}}}}_c^4 = \underline{\underline{0}}$, and the expression for the exponential reduces to

$$\exp(\bar{\underline{\underline{\mathcal{H}}}}_c x) = \underline{\underline{I}} + x\bar{\underline{\underline{\mathcal{H}}}}_c + \frac{x^2}{2!}\bar{\underline{\underline{\mathcal{H}}}}_c^2 + \frac{x^3}{3!}\bar{\underline{\underline{\mathcal{H}}}}_c^3. \tag{80}$$

The general form of the exponential is given by eq. (61); the motion tensor is given by eq. (79) and matrix $\underline{\underline{\mathcal{B}}}$ reduces to

$$\underline{\underline{\mathcal{B}}} = x\bar{\underline{\underline{S}}}^* + \frac{x^2}{2!}(\bar{\underline{\underline{S}}}^*\tilde{\bar{\underline{\underline{\mathcal{K}}}}}^{*T} - \tilde{\bar{\underline{\underline{\mathcal{K}}}}}^*\bar{\underline{\underline{S}}}^*) - \frac{x^3}{3!}\tilde{\bar{\underline{\underline{\mathcal{K}}}}}^*\bar{\underline{\underline{S}}}^*\tilde{\bar{\underline{\underline{\mathcal{K}}}}}^{*T}. \tag{81}$$

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