Tensorial Parameterization of Rotation and Motion

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Abstract

The parameterization of rotation and motion is the subject of continuous research and development in many theoretical and applied fields of mechanics such as rigid body, structural, and multibody dynamics. Tensor analysis expresses the invariance of the laws of physics with respect to change of basis and change of frame operations. Consequently, it is imperative to formulate mechanics problems in terms of tensorial quantities. This paper presents a formal proof that rotation and motion parameters vectors are tensors if and only if they are parallel to the eigenvectors of the rotation and motion tensors, respectively, associated with their unit eigenvalues. The tensorial nature and properties of the resulting vectorial parameterization of rotation and motion are investigated in details and explicit expressions are given for the rotation and motion tensors, as well as the corresponding tangent tensors. Several important tensor identities are shown to hold for all vectorial parameterization of rotation and motion.

1 Introduction

The description of finite rotations, simply called rotations in this paper, is a fundamental tool used in dynamics, multibody dynamics, and mechanics. Numerous techniques presenting various properties and advantages have been proposed to parameterize rotations, and exhaustive reviews of these techniques may be found in refs. [1, 2, 3, 4].

Whether originating from geometric, algebraic, or matrix approaches, parameterizations of rotation are most naturally categorized into two classes: *vectorial* and *non-vectorial* parameterizations. For vectorial parameterizations, a rotation parameter vector is defined, whose components transform according to the rules of transformation for first order tensors under a change of basis operation. Bauchau and Trainelli [5] have introduced the *vectorial parameterization of rotation* by selecting the rotation parameter vector to be parallel to the eigenvector of the rotation tensor associated with its unit eigenvalue.

The description of finite motion, simply called motion in this paper, is an equally important topic used in dynamics, multibody dynamics, and mechanics. Here again, a distinction must be made between vectorial and non-vectorial parameterizations of motion. For vectorial parameterizations, a motion parameter vector is defined, which combines the components of two vectors, a displacement related parameter vector and a rotation related parameter vector. The components of each vector transform according to the rules of transformation for first order tensors under a change of basis operation. Bauchau and Choi [6] introduced the *vectorial parameterization of motion* by selecting the motion parameter vector to be parallel to the eigenvector of the motion tensor associated with its unit eigenvalue.

Tensor analysis expresses the invariance of the laws of physics with respect to change of basis and change of frame operations [7]. Consequently, it is imperative to formulate mechanics problems in terms of tensorial quantities. In references [5] and [6], the rotation and motion parameter vectors were chosen to be parallel to the eigenvector of the rotation and motion tensors, respectively. Although this approach yields interesting properties for the parameter vectors, this choice seems to be rather arbitrary.

This paper presents a formal proof of the equivalence between the two fundamental properties of the vectorial parameterizations of rotation and motion: their tensorial nature and their parallelism to the eigenvector of the rotation and motion tensors, respectively. In other words, selecting the rotation and motion parameters to be parallel to the eigenvector of the rotation and motion tensors, respectively, is not an arbitrary choice, but rather, is the only choice that leads to tensorial parameterizations of rotation and motion, respectively.

The rotation and motion tensors are *intrinsic*, *i.e.*, they are independent of the parametrization used to represent them. Furthermore, they are second order tensors. The rotation and motion parameter vectors are no intrinsic (they are "parameters"), but they form first order tensors when the vectorial parameterization of rotation or motion is used.

The angular velocity vector, $\underline{\omega}$, is a nonholonomic vector, *i.e.*, there exist no vector, say $\underline{x}$, such that $\underline{\omega} = \dot{\underline{x}}$. Rather, if $\underline{q}$ denotes a set of three rotation parameters, $\underline{\omega} = \underline{\underline{H}}(\underline{q})\dot{\underline{q}}$, where $\underline{\underline{H}}$ is the tangent operator that depends on the rotation parameters. Because it depends on the selected parameterization explicitly, the tangent operator is not intrinsic, but it is a second order tensor when the vectorial parameterization of rotation is used. These results generalize for the case of motion, provided that the vectorial parameterization of motion is used.

Because this paper focuses on the tensorial nature of the vectorial parameterizations of rotation and motion, expressions are given for the rotation and motion tensors that explicitly exhibit their tensorial nature. Generic expressions are also given for the tangent tensor of the vectorial parameterization of both rotation and motion. These generic expressions apply to all vectorial parameterizations of rotation and motion. Finally, it is shown that many of the tensor identities related to rotation operations also apply, *mutatis mutandis*, to motion operations.

Section 2 reviews the fundamental properties of rotation, and leads to the formal proof that tensorial parameterizations must feature rotation parameter vectors parallel to the eigenvector of the rotation tensor, presented in section 3. Finally, explicit expression for the rotation and tangent tensors are given in section 3.2, which also states important properties of these tensors. A similar pattern is followed in sections 4, 5, and 5.4 for the vectorial parameterization of motion.

2 Fundamental properties of rotation

Euler's theorem [8] on rotations states that *any arbitrary rotation that leaves a point fixed can be viewed as a single rotation of magnitude ϕ about a unit vector $\bar{n}$.* Consider an arbitrary vector $\underline{a}$, and let the rotation of magnitude ϕ about unit vector $\bar{n}$ bring this vector to $\underline{b}$, as depicted in fig. 1. The *rotation tensor*, $\underline{\underline{R}}$, relates these two vectors, $\underline{b} = \underline{\underline{R}}\,\underline{a}$, and can be determined from basic geometric arguments to be

$$\underline{\underline{R}} = \underline{\underline{I}} + \sin\phi\ \widetilde{n} + (1 - \cos\phi)\widetilde{n}\widetilde{n}, \tag{1}$$

where $\underline{\underline{I}}$ indicates the unit tensor. This result is known as *Euler's rotation formula* [9]. Notation $\widetilde{a}$ indicates the second order, skew-symmetric tensor constructed from the component of vector $\underline{a}$,

$$\widetilde{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{2}$$

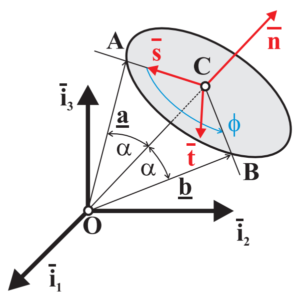

Figure 1: A rotation of magnitude ϕ about axis $\bar{n}$.

The first fundamental property of the rotation tensor is that *it possesses a unit eigenvalue, $\lambda = +1$, associated with eigenvector $\bar{n}$.* Indeed, it follows from eq. (1) that

$$\underline{\underline{R}}\,\bar{n} = \bar{n}. \tag{3}$$

The second fundamental property of the rotation tensor is that it is an *orthogonal tensor*. Using eq. (1), it is readily verified that $\underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{I}}$. The rotation tensor belongs to the class of *proper orthogonal tensors* for which $\det(\underline{\underline{R}}) = +1$.

If $\underline{\underline{R}}_2$ is an orthogonal tensor, the following equivalence can be proved based on simple algebraic manipulations,

$$\widetilde{a}_3 = \underline{\underline{R}}_2^T \widetilde{a}_1 \underline{\underline{R}}_2 \Leftrightarrow \underline{a}_3 = \underline{\underline{R}}_2^T \underline{a}_1. \tag{4}$$

Let $\underline{a}_1$ and $\underline{a}_3$ be the components of vector $\underline{a}$ resolved in two bases, $\mathcal{B}_1$ and $\mathcal{B}_3$, respectively. Furthermore, let $\underline{\underline{R}}_2$ be the components of the rotation tensor that brings basis $\mathcal{B}_1$ to $\mathcal{B}_3$, resolved in basis $\mathcal{B}_1$. The right-hand side of equivalence (4) now expresses the rules of transformation for the components of the first order tensor, $\underline{a}$, under a change of basis operation, and the left-hand side of the same equivalence expresses the rules of transformation for the components of the second order tensor, $\widetilde{a}$, under the same change of basis operation.

The angular velocity vector, $\underline{\omega}$, is defined as $\widetilde{\omega} = \dot{R}R^T$ and can be expressed in terms of the quantities $\phi(t)$ and $\bar{n}(t)$ that characterize the rotation. Introducing the rotation tensor, eq. (1), and using vector identities yields

$$\underline{\omega} = \dot{\phi}\ \bar{n} + \sin\phi\ \dot{\bar{n}} + (1-\cos\phi)\widetilde{\bar{n}}\dot{\bar{n}} \tag{5}$$

3 The vectorial parameterization of rotation

The *vectorial parameterization of rotation* consists of a minimum set of three parameters defining the components of a *rotation parameter vector*. Bauchau and Trainelli [5] introduced this parameterization by selecting the rotation parameter vector to be parallel to the eigenvector of the rotation tensor associated with its unit eigenvalue. While this parallelism is an important property of the vectorial parameterization of rotation, its tensorial nature is equally important and is explored in the following sections.

3.1 Fundamental properties

Consider three rotations of magnitudes ϕ_1, ϕ_2, and ϕ_3, about unit vectors $\bar{n}_1$, $\bar{n}_2$, and $\bar{n}_3$, respectively. The three rotations, denoted $(\phi_1, \bar{n}_1)$, $(\phi_2, \bar{n}_2)$, and $(\phi_3, \bar{n}_3)$, respectively, are associated with three rotation tensors, denoted $\underline{\underline{R}}_1$, $\underline{\underline{R}}_2$, and $\underline{\underline{R}}_3$, respectively, through Euler's formula, eq. (1).

Assume that the following triple product of rotation tensors relates these three quantities,

$$\underline{\underline{R}}_3 = \underline{\underline{R}}_2^T \underline{\underline{R}}_1 \underline{\underline{R}}_2. \tag{6}$$

This operation corresponds to a change of basis for second order tensors: $\underline{\underline{R}}_1$ and $\underline{\underline{R}}_3$ are the components of the same rotation tensor expressed in two bases related by rotation tensor $\underline{\underline{R}}_2$.

Using Euler's formula, eq. (1), eq. (6) now becomes

$$\underline{\underline{R}}_3 = \underline{\underline{I}} + \sin\phi_1 \widetilde{\underline{\underline{R}}_2^T \bar{n}_1} + (1-\cos\phi_1) \widetilde{\underline{\underline{R}}_2^T \bar{n}_1} \widetilde{\underline{\underline{R}}_2^T \bar{n}_1},$$

where eq. (4) was used. Comparing this result with Euler's formula implies that

$$\phi_3 = \phi_1, \tag{7a}$$

$$\bar{n}_3 = \underline{\underline{R}}_2^T \bar{n}_1. \tag{7b}$$

These equations express the two conditions required for the proper transformation of rotation tensors components under a change of basis.

Let $p(\phi)$ be an arbitrary odd function of angle ϕ; eq. (7a) then implies $p(\phi_3) = p(\phi_1)$. Multiplication of eq. (7b) by $p(\phi_3)$ on the left-hand side and $p(\phi_1) = p(\phi_3)$ on the right-hand side then yields

$$p(\phi_3)\bar{n}_3 = \underline{\underline{R}}_2^T\ p(\phi_1)\bar{n}_1. \tag{8}$$

This equation is equivalent to eqs. (7). Indeed, premultiplying the left-hand side of eq. (8) by $p(\phi_3)\bar{n}_3^T$ and the right-hand side by $p(\phi_1)\bar{n}_1^T \underline{\underline{R}}_2$ yields $p^2(\phi_3) = p^2(\phi_1)$, or $p(\phi_3) = \pm p(\phi_1)$, since $\bar{n}_1$ and $\bar{n}_3$ are unit vectors and $\underline{\underline{R}}_2$ an orthogonal tensor. Because p is an odd function of ϕ, it follows that $\phi_3 = \pm\phi_1$, and eq. (8) then yields $\bar{n}_3 = \pm \underline{\underline{R}}_2^T \bar{n}_1$. These two solutions, $\phi_3 = +\phi_1$, $\bar{n}_3 = +\underline{\underline{R}}_2^T \bar{n}_1$, and $\phi_3 = -\phi_1$, $\bar{n}_3 = -\underline{\underline{R}}_2^T \bar{n}_1$, correspond to identical rotations, both defining eqs. (7).

The *vectorial parametrization of rotation* is defined as

$$\underline{p} = p(\phi)\bar{n}, \tag{9}$$

where $\underline{p}$ is the rotation parameter vector. Equation (8) can now be recast in a more compact manner as

$$\underline{p}_3 = \underline{\underline{R}}^T(\underline{p}_2)\ \underline{p}_1. \tag{10}$$

The discussion presented above establishes that the tensorial nature of the rotation tensor expressed by the transformation rule of its components, eq. (6), implies the tensorial nature of the rotation parameter vector expressed by the transformation rule of its components, eq. (10). It is easily shown that the process can be reversed, *i.e.*, tensorial nature of the rotation parameter vector implies that of the rotation tensor.

In summary, the vectorial parameterization of rotation presents two fundamental properties.

(1) The vectorial parameterization of rotation is tensorial in nature, as expressed by the following equivalence,

$$\underline{\underline{R}}(\underline{p}_3) = \underline{\underline{R}}^T(\underline{p}_2)\underline{\underline{R}}(\underline{p}_1)\underline{\underline{R}}(\underline{p}_2) \Leftrightarrow \underline{p}_3 = \underline{\underline{R}}^T(\underline{p}_2)\underline{p}_1. \tag{11}$$

The tensorial nature of the second order rotation tensor implies and is implied by the tensorial nature of the rotation parameter vector, a first order tensor.

(2) Rotation parameter vectors are parallel to the *eigenvector of the rotation tensor* corresponding to its unit eigenvalue. Because unit vector $\bar{n}$ is the eigenvector of the rotation tensor associated with its unit eigenvalue, eq. (3), the definition of the rotation parameter vector, eq. (9), implies its parallelism to $\bar{n}$.

Because these two properties imply each other, either can be taken as the definition of the vectorial parameterization of rotation. A parameterization of rotation is tensorial if and only if the rotation parameter vector is parallel to the eigenvector of the rotation tensor associated with its unit eigenvalue.

The rotation parameter vector is not yet fully defined because function $p(\phi)$, called the *generating function*, is still arbitrary. Generating functions must be odd functions of the rotation angle, ϕ, and present the following limit behavior

$$\lim_{\phi\to 0} p(\phi) = \phi, \tag{12}$$

i.e., all rotation parameter vectors must approach the *infinitesimal rotation vector* when $\phi \to 0$.

Because it is imperative to formulate mechanics problems in terms of tensors, the vectorial parameterization of rotation should be the preferred choice. Indeed, it is the only parameterization to yield rotation parameter vectors that are first order tensors.

3.2 Basic operators

The explicit expression of the rotation tensor in term of the vectorial parameterization is easily obtained from Euler's formula, eq. (1),

$$\underline{\underline{R}} = \underline{\underline{I}} + \zeta_1(\phi)\ \widetilde{p} + \zeta_2(\phi)\ \widetilde{p}\widetilde{p}, \tag{13}$$

where $\zeta_1(\phi)$ and $\zeta_2(\phi)$ are even functions of the rotation angle, ϕ, defined as

$$\zeta_1(\phi) = \frac{\sin\phi}{p} = \nu\cos\frac{\phi}{2} = \frac{\nu^2}{\varepsilon}, \tag{14a}$$

$$\zeta_2(\phi) = \frac{1-\cos\phi}{p^2} = \frac{\nu^2}{2} = \frac{\varepsilon\zeta_1}{2}. \tag{14b}$$

Two even functions of the rotation angle play an important role in the vectorial parameterization of rotation,

$$\nu = \frac{2\sin\phi/2}{p}, \tag{15a}$$

$$\varepsilon = \frac{2\tan\phi/2}{p} = \frac{\nu}{\cos\phi/2}. \tag{15b}$$

Functions $p(\phi)$, $\zeta_1(\phi)$, and $\zeta_2(\phi)$ solely depend on the magnitude, ϕ, of the rotation. Because this angle is invariant under a change of basis, these functions are also invariant under a change of basis, and hence, are zeroth order tensors. Since $\widetilde{p}$ is a second order tensor, see eq. (4), eq. (13) proves that the rotation tensor is a second order tensor because it is obtained through tensor operations from zeroth and second order tensors.

An important multiplicative decomposition of the rotation tensor is $\underline{\underline{R}} = \underline{\underline{G}}\,\underline{\underline{G}}$, where the half-angle rotation tensor is

$$\underline{\underline{G}} = \underline{\underline{I}} + \frac{\nu}{2}\widetilde{p} + \frac{1-\cos\phi/2}{p^2}\widetilde{p}\widetilde{p}, \tag{16}$$

A second multiplicative decomposition of the rotation tensor is

$$\underline{\underline{R}} = (\underline{\underline{I}} + \frac{\varepsilon}{2}\widetilde{p})(\underline{\underline{I}} - \frac{\varepsilon}{2}\widetilde{p})^{-1} = (\underline{\underline{I}} - \frac{\varepsilon}{2}\widetilde{p})^{-1}(\underline{\underline{I}} + \frac{\varepsilon}{2}\widetilde{p}). \tag{17}$$

Furthermore,

$$(\underline{\underline{I}} - \frac{\varepsilon}{2}\widetilde{p})^{-1} = \frac{\underline{\underline{R}} + \underline{\underline{I}}}{2}. \tag{18}$$

Taking a time derivative of the rotation parameter vector yields $\dot{\underline{p}} = p'\dot{\phi}\bar{n} + p\dot{\bar{n}}$, where $p' = \mathrm{d}p/\mathrm{d}\phi$ and the notation $(\cdot)'$ indicates a derivative with respect to ϕ. Vector identities then lead to $\widetilde{n}\widetilde{n}\dot{\underline{p}} = p\widetilde{n}\widetilde{n}\dot{\bar{n}} = -p\dot{\bar{n}} = p'\dot{\phi}\bar{n} - \dot{\underline{p}}$, because $\bar{n}$ is a unit vector and hence, $\bar{n}^T\dot{\bar{n}} = 0$. Introducing these results into the expression for the angular velocity, eq. (5), then leads to

$$\underline{\omega} = \underline{\underline{H}}(\underline{p})\dot{\underline{p}}. \tag{19}$$

The tangent operator, $\underline{\underline{H}}(\underline{p})$, is given by

$$\underline{\underline{H}}(\underline{p}) = \sigma_0(\phi)\ \underline{\underline{I}} + \sigma_1(\phi)\ \widetilde{p} + \sigma_2(\phi)\ \widetilde{p}\widetilde{p}, \tag{20}$$

where $\sigma_0(\phi)$, $\sigma_1(\phi)$, and $\sigma_2(\phi)$ are even functions of the rotation angle, ϕ, defined as

$$\sigma_0(\phi) = \frac{1}{p'}, \tag{21a}$$

$$\sigma_1(\phi) = \frac{1-\cos\phi}{p^2} = \zeta_2, \tag{21b}$$

$$\sigma_2(\phi) = \frac{\sigma_0 - \zeta_1}{p^2}. \tag{21c}$$

These three functions are zeroth order tensors because they are functions of angle ϕ, which is invariant under a change of basis.

The inverse of this operator is

$$\underline{\underline{H}}^{-1}(\underline{p}) = \chi_0\underline{\underline{I}} - \frac{1}{2}\widetilde{p} + \chi_2\widetilde{p}\widetilde{p}, \tag{22}$$

where $\chi_0(\phi)$ and $\chi_2(\phi)$ are even functions of the rotation angle, ϕ, defined as

$$\chi_0(\phi) = p', \tag{23a}$$

$$\chi_2(\phi) = \frac{1}{p^2}\left(p' - \frac{1}{\varepsilon}\right). \tag{23b}$$

The tangent operator, $\underline{\underline{H}}$, enjoys the following remarkable properties,

$$\underline{\underline{R}} = \underline{\underline{H}}\,\underline{\underline{H}}^{-T} = \underline{\underline{H}}^{-T}\underline{\underline{H}}, \tag{24a}$$

$$\underline{\underline{R}} - \underline{\underline{I}} = \widetilde{p}\underline{\underline{H}} = \underline{\underline{H}}\,\widetilde{p}, \tag{24b}$$

$$\underline{\underline{I}} - \underline{\underline{R}}^T = \nu^2\widetilde{p}\underline{\underline{H}}^{-1} = \nu^2\underline{\underline{H}}^{-1}\widetilde{p}, \tag{24c}$$

$$\widetilde{p} = \underline{\underline{H}}^{-T} - \underline{\underline{H}}^{-1}. \tag{24d}$$

The tangent operator is specific to a particular vectorial parameterization, *i.e.*, its expression depends on the choice of the generating function. It is, however, a second order tensor. Using the definition of this operator, eq. (20), the following result is easily established,

$$\underline{\underline{H}}(\underline{p}_3) = \underline{\underline{R}}^T(\underline{p}_2)\underline{\underline{H}}(\underline{p}_1)\underline{\underline{R}}(\underline{p}_2) \Leftrightarrow \underline{p}_3 = \underline{\underline{R}}^T(\underline{p}_2)\underline{p}_1. \tag{25}$$

Although the tangent tensor is not an intrinsic tensor because it depends on the choice of a specific generating function, it is a second order tensor for all vectorial parameterizations of rotation.

All the properties presented in this section are valid for all vectorial parameterization of rotation. Specific choices of the generating function lead to well-know parameterizations such as the exponential map of rotation, the Cayley-Gibbs-Rodrigues parameterization, or the Wiener-Milenković parameterization. More details are found in reference [5, 10].

4 Fundamental properties of motion

Mozzi-Chasles' theorem [11, 12] states that *the most general motion of a rigid body consists of a translation along an axis $\bar{n}$ followed by a rotation about that same axis.* In contrast with the rotation operation described in the previous sections, the motion of a rigid body is characterized by both translation and rotation. Frames $\mathcal{F}^I = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$ and $\mathcal{F} = \left[\mathbf{A}, \mathcal{B}^* = (\bar{b}_1, \bar{b}_2, \bar{b}_3)\right]$ are attached to the rigid body in its reference and final configurations, respectively.

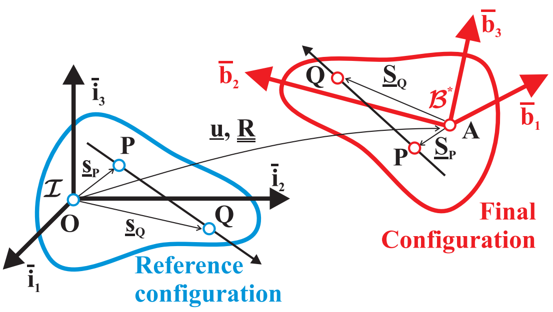

Figure 2: A line of a rigid body in the reference and final configurations.

Consider a material line, **PQ**, of a rigid body, as depicted in fig. 2. The Plücker coordinates of this material line evaluated with respect to point **A** and resolved in basis $\mathcal{B}^*$ are denoted $\underline{\mathcal{Q}}^*$, and $\underline{\mathcal{Q}}$ denotes the Plücker coordinates of the same material line evaluated with respect to point **O** and resolved in basis $\mathcal{I}$. The relationship between these two sets of Plücker coordinates is $\underline{\mathcal{Q}} = \underline{\underline{\mathcal{C}}}\,\underline{\mathcal{Q}}^*$, where the *motion tensor* [13, 10] and its inverse are defined as

$$\underline{\underline{\mathcal{C}}}(\underline{u}, \underline{\underline{R}}) = \begin{bmatrix} \underline{\underline{R}} & \widetilde{u}\underline{\underline{R}} \\ 0 & \underline{\underline{R}} \end{bmatrix}, \tag{26a}$$

$$\underline{\underline{\mathcal{C}}}^{-1}(\underline{u}, \underline{\underline{R}}) = \begin{bmatrix} \underline{\underline{R}}^T & \underline{\underline{R}}^T\widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{R}}^T \end{bmatrix}. \tag{26b}$$

The displacement of the reference point on the rigid body is denoted $\underline{u}$ and $\underline{\underline{R}}$ is the rotation tensor that defines the final orientation of the rigid body.

4.1 Intrinsic expression of the motion tensor

The motion tensor was defined by eq. (26a), which is not an intrinsic expression because the displacement vector of the reference point of the rigid body, $\underline{u}$, explicitly appears in this definition. In this section, an intrinsic expression of the motion tensor is sought in which vector $\underline{u}$ does not appear explicitly.

It seems logical to express the displacement vector of the reference point of the rigid body with as $\underline{u} = d\bar{n} - (\underline{\underline{R}} - \underline{\underline{I}})\underline{s}_Q$, where $d = \bar{n}^T\underline{u}$ is the intrinsic displacement of the rigid body and $\underline{s}_Q = \widetilde{n}\underline{\underline{G}}^T\underline{u}/(2\sin\phi/2)$ the position vector of an arbitrary point of the Mozzi-Chasles axis. Introducing eq. (16) leads to $\underline{u} = d\bar{n} - 2\sin\phi/2\; \underline{\underline{G}}\widetilde{n}\underline{s}_Q$, and finally

$$\begin{aligned} \widetilde{u}\underline{\underline{R}} &= d\widetilde{n}\underline{\underline{R}} - 2\sin\frac{\phi}{2}\,\widetilde{\underline{\underline{G}}\widetilde{n}\underline{s}_Q}\,\underline{\underline{R}} \\ &= d\cos\phi\,\widetilde{n} + d\sin\phi\,\widetilde{n}\widetilde{n} - 2\sin\frac{\phi}{2}\,\widetilde{\underline{\underline{G}}\widetilde{n}\underline{s}_Q}\,\underline{\underline{R}}. \end{aligned} \tag{27}$$

This equation now relates the product $\widetilde{u}\underline{\underline{R}}$ to the intrinsic displacement of the body and to the position vector of an arbitrary point of the Mozzi-Chasles axis, $\underline{s}_Q$.

4.1.1 The displacement related vector

A vector, $\underline{m}$, related to the displacement of the rigid body is now introduced, $\underline{m} = -\widetilde{n}\underline{s}_Q$; given the displacement of the body, the position vector, $\underline{s}_Q$, of a point of the Mozzi-Chasles axis can be evaluated to yield $\underline{m}$. The inverse operation, however, is not possible because $\widetilde{n}$ is a singular tensor.

To remedy this situation and enable a one-to-one mapping between vectors $\underline{m}$ and $\underline{u}$, the displacement related vector is modified to become $\underline{m} = -\widetilde{n}\underline{s}_Q + \lambda\bar{n}$, where λ is an arbitrary scalar. Because $\underline{m}$ is a displacement related vector, scalar λ is chosen to be proportional to the intrinsic displacement of the body

$$\lambda = \frac{\alpha d}{2\sin\phi/2}, \tag{28}$$

where $\alpha(\phi)$ is an arbitrary function of the rotation angle, ϕ.

The displacement related vector, $\underline{m}$, now becomes,

$$\underline{m} = -\widetilde{n}\underline{s}_Q + \lambda\bar{n} = -\frac{\widetilde{n}\widetilde{n}\underline{\underline{G}}^T}{2\sin\phi/2}\underline{u} + \frac{\alpha}{2\sin\phi/2}\bar{n}\bar{n}^T\underline{u}. \tag{29}$$

Use of vector identities now yields

$$\underline{m} = \underline{\underline{E}}(\phi)\underline{u}, \tag{30}$$

where the second-order tensor, $\underline{\underline{E}}(\phi)$, is defined as

$$\underline{\underline{E}}(\phi) = \frac{\alpha}{2\sin\phi/2}\underline{\underline{I}} - \frac{1}{2}\widetilde{n} + \left[\frac{\alpha}{2\sin\phi/2} - \frac{1}{2\tan\phi/2}\right]\widetilde{n}\widetilde{n}. \tag{31}$$

4.1.2 Properties of the generalized vector product operator

The generalized vector product operator is defined as

$$\underline{\underline{\mathcal{W}}}(\underline{\mathcal{N}}) = \begin{bmatrix} \widetilde{n} & \widetilde{m} \\ \underline{\underline{0}} & \widetilde{n} \end{bmatrix}, \tag{32}$$

where vector $\underline{\mathcal{N}}$ combines the displacement related vector, $\underline{m}$, and the axis about which the rotation is taking place, $\bar{n}$,

$$\underline{\mathcal{N}} = \begin{Bmatrix} \underline{m} \\ \bar{n} \end{Bmatrix}. \tag{33}$$

The generalized vector product operator enjoys remarkable properties that generalize those of the skew-symmetric operator.

First, the skew-symmetric operator, $\widetilde{n}$, possesses a null eigenvalue, $\widetilde{n}\bar{n} = 0\bar{n}$. Similarly, the generalized vector product operator possesses a null eigenvalue with a multiplicity of two. Two linearly independent eigenvectors associated with this null eigenvalue are

$$\underline{\mathcal{N}}_1^\dagger = \begin{Bmatrix} \bar{n} \\ \underline{0} \end{Bmatrix}, \quad \underline{\mathcal{N}}_2^\dagger = \begin{Bmatrix} \dfrac{\underline{\underline{G}}^T\underline{u}}{2\sin\phi/2} \\ \bar{n} \end{Bmatrix}, \tag{34}$$

The fact that $\underline{\mathcal{N}}_1^\dagger$ is an eigenvector of the generalized vector product operator stems from the corresponding property for the skew-symmetric operator, $\widetilde{n}\bar{n} = 0\bar{n}$. It is readily verified that $\underline{\mathcal{N}}_2^\dagger$ is also an eigenvector of the generalized vector product operator, indeed, $\widetilde{n}\,\underline{\underline{G}}^T\underline{u}/(2\sin\phi/2) + \widetilde{m}\bar{n} = \underline{s}_Q - \widetilde{n}(-\widetilde{n}\underline{s}_Q - \lambda\bar{n}) = 0$.

The second property of the generalized vector product operator generalizes the behavior of the skew-symmetric operator under a change of basis operation, eq. (4). Consider the following triple matrix product

$$\begin{bmatrix} \widetilde{n}_3 & \widetilde{m}_3 \\ \underline{\underline{0}} & \widetilde{n}_3 \end{bmatrix} = \begin{bmatrix} \underline{\underline{R}}_2^T & \underline{\underline{R}}_2^T\widetilde{u}_2^T \\ \underline{\underline{0}} & \underline{\underline{R}}_2^T \end{bmatrix} \begin{bmatrix} \widetilde{n}_1 & \widetilde{m}_1 \\ \underline{\underline{0}} & \widetilde{n}_1 \end{bmatrix} \begin{bmatrix} \underline{\underline{R}}_2 & \widetilde{u}_2\underline{\underline{R}}_2 \\ \underline{\underline{0}} & \underline{\underline{R}}_2 \end{bmatrix}.$$

This equality implies two conditions. The first condition is $\widetilde{n}_3 = \underline{\underline{R}}_2^T\widetilde{n}_1\underline{\underline{R}}_2$, and, in view of eq. (4), results in $\bar{n}_3 = \underline{\underline{R}}_2^T\bar{n}_1$. The second condition is $\widetilde{m}_3 = \underline{\underline{R}}_2^T(\widetilde{m}_1 + \widetilde{n}_1\widetilde{u}_2 - \widetilde{u}_2\widetilde{n}_1)\underline{\underline{R}}_2$, and tensor identities then lead to $\underline{m}_3 = \underline{\underline{R}}_2^T(\underline{m}_1 + \widetilde{n}_1\underline{u}_2)$. The results can be summarized by the following equivalence,

$$\underline{\underline{\mathcal{W}}}(\underline{\mathcal{N}}_3) = \underline{\underline{\mathcal{C}}}_2^{-1}\underline{\underline{\mathcal{W}}}(\underline{\mathcal{N}}_1)\underline{\underline{\mathcal{C}}}_2 \Leftrightarrow \underline{\mathcal{N}}_3 = \underline{\underline{\mathcal{C}}}_2^{-1}\underline{\mathcal{N}}_1. \tag{35}$$

The third property of the generalized vector product operator generalizes the following identity, $\widetilde{n}\widetilde{n}\widetilde{n} + \widetilde{n} = 0$, which holds for any unit vector, $\bar{n}$,

$$\underline{\underline{\mathcal{W}}}^3(\underline{\mathcal{N}}) + \underline{\underline{\mathcal{Z}}}(2\lambda, 1)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{N}}) = \underline{\underline{0}}, \tag{36}$$

where $\lambda = \bar{n}^T\underline{m}$. In eq. (36), tensor $\underline{\underline{\mathcal{Z}}}$, a function of two scalars, α and β, is defined as

$$\underline{\underline{\mathcal{Z}}}(\alpha, \beta) = \begin{bmatrix} \beta\underline{\underline{I}} & \alpha\underline{\underline{I}} \\ \underline{\underline{0}} & \beta\underline{\underline{I}} \end{bmatrix}. \tag{37}$$

4.1.3 Intrinsic expression of the motion tensor

To obtain an intrinsic expression of the motion tensor, eq. (27) is now recast in terms of the displacement related vector, $\underline{m}$. Using eq. (16), $\underline{\underline{G}}\widetilde{n} = \cos\phi/2\,\widetilde{n} + \sin\phi/2\,\widetilde{n}\widetilde{n}$; it then follows that $-\underline{\underline{G}}\widetilde{n}\underline{s}_Q = \cos\phi/2(\underline{m} - \lambda\bar{n}) + \sin\phi/2\,\widetilde{n}\underline{m}$, and finally, $-2\sin\phi/2\,\underline{\underline{G}}\widetilde{n}\underline{s}_Q = \sin\phi(\underline{m} - \lambda\bar{n}) + (1-\cos\phi)\,\widetilde{n}\underline{m}$. Using Euler's formula to express the rotation tensor, eq. (27) becomes

$$\begin{aligned}\widetilde{u}\underline{\underline{R}} = {} & d\cos\phi\,\widetilde{n} + d\sin\phi\,\widetilde{n}\widetilde{n} + \sin\phi\,\widetilde{m} + (1-\cos\phi) \\ & (\widetilde{n}\widetilde{m} + \widetilde{m}\widetilde{n}) - \lambda\sin\phi\,\widetilde{n} - 2\lambda(1-\cos\phi)\widetilde{n}\widetilde{n} \\ = {} & \sin\phi\,\widetilde{m} + dc_1\widetilde{n} + (1-\cos\phi)\,(\widetilde{n}\widetilde{m} + \widetilde{m}\widetilde{n}) + dc_2\widetilde{n}\widetilde{n},\end{aligned}$$

where eq. (28) was used to eliminate λ, and coefficients c_1 and c_2 are defined as

$$c_1 = \cos\phi - \alpha(\phi)\cos\phi/2, \tag{38a}$$

$$c_2 = \sin\phi - 2\alpha(\phi)\sin\phi/2. \tag{38b}$$

The motion tensor, eq. (26a), and its inverse now become

$$\begin{aligned}\underline{\underline{\mathcal{C}}}(\underline{\mathcal{N}}) = {} & \underline{\underline{\mathcal{I}}} + \underline{\underline{\mathcal{Z}}}(dc_1, \sin\phi)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{N}}) \\ & + \underline{\underline{\mathcal{Z}}}(dc_2, 1-\cos\phi)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{N}})\underline{\underline{\mathcal{W}}}(\underline{\mathcal{N}}),\end{aligned} \tag{39a}$$

$$\begin{aligned}\underline{\underline{\mathcal{C}}}^{-1}(\underline{\mathcal{N}}) = {} & \underline{\underline{\mathcal{I}}} - \underline{\underline{\mathcal{Z}}}(dc_1, \sin\phi)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{N}}) \\ & + \underline{\underline{\mathcal{Z}}}(dc_2, 1-\cos\phi)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{N}})\underline{\underline{\mathcal{W}}}(\underline{\mathcal{N}}).\end{aligned} \tag{39b}$$

The parallel between this intrinsic expression for the motion tensor and that for the rotation tensor given by Euler's formula, eq. (1), is striking. Clearly, the skew-symmetric tensor, $\widetilde{n}$, appearing in the expression for the rotation tensor is generalized by tensor $\underline{\underline{\mathcal{W}}}$, appearing in that for the motion tensor. The two scalars, $\sin\phi$ and $(1-\cos\phi)$, appearing in the expression for the rotation tensor becomes the second arguments of operator $\underline{\underline{\mathcal{Z}}}$ appearing in that for the motion tensor.

Euler's formula, eq. (1), provides an intrinsic expression for the rotation tensor and is a direct consequence of Euler's theorem on rotation. The intrinsic expression for the motion tensor is based on the location of the Mozzi-Chasles axis, and hence, is a direct consequence of the Mozzi-Chasles theorem.

4.2 Properties of the motion tensor

The motion tensor can be factorized in the following manner

$$\underline{\underline{\mathcal{C}}} = \begin{bmatrix} \underline{\underline{I}} & \widetilde{u} \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix} \begin{bmatrix} \underline{\underline{R}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{R}} \end{bmatrix} = \underline{\underline{\mathcal{T}}}\,\underline{\underline{\mathcal{R}}}, \tag{40}$$

where $\underline{\underline{\mathcal{R}}}$ is the *rotation operator* and $\underline{\underline{\mathcal{T}}}$ the *translation operator*. The eigenvalues of the motion tensor are now easily computed. Indeed, $\det(\underline{\underline{\mathcal{C}}}) = \det(\underline{\underline{\mathcal{T}}})\det(\underline{\underline{\mathcal{R}}}) = \det(\underline{\underline{\mathcal{T}}})\det^2(\underline{\underline{R}})$ and because $\det(\underline{\underline{\mathcal{T}}}) = 1$, it follows that $\det(\underline{\underline{\mathcal{C}}}) = \det^2(\underline{\underline{R}})$. Hence, the eigenvalues of the motion tensor are identical to those of the rotation tensor, but each with a multiplicity of two. The motion tensor, however, unlike the rotation tensor, is not an orthogonal tensor.

The intrinsic expression of the motion tensor, eq. (39a), shows that the eigenvectors of the generalized vector product operator associated with its null eigenvalue are identical to the eigenvectors of the motion tensor associated with its unit eigenvalue. It follows that two linearly independent eigenvectors of the motion tensor associated with its unit eigenvalues are given by eq. (34).

Any linear combination of eigenvectors $\underline{\mathcal{N}}_1^\dagger$ and $\underline{\mathcal{N}}_2^\dagger$ is still an eigenvector of the motion tensor. For instance, vector $\underline{\mathcal{N}}$, defined by eq. (33), can be expressed as follows

$$\underline{\mathcal{N}} = \frac{(\alpha-1)d}{2\sin\phi/2}\underline{\mathcal{N}}_1^\dagger + \underline{\mathcal{N}}_2^\dagger. \tag{41}$$

This means that vector $\underline{\mathcal{N}}$, the key to the intrinsic expression of the motion tensor, is an eigenvector of the motion tensor, for any value of α.

4.3 Frame change for the incremental motion vector

Consider the incremental displacement of point **A** shown in fig 2. The components of this incremental displacement vector resolved in bases $\mathcal{I}$ and $\mathcal{B}^*$ are denoted $\mathrm{d}\underline{u}$ and $\underline{\underline{R}}^T\mathrm{d}\underline{u}$, respectively. The components of the incremental rotation vector of the body are denoted $\underline{\mathrm{d}\psi} = \mathrm{axial}(\mathrm{d}\underline{\underline{R}}\,\underline{\underline{R}}^T)$ and $\underline{\mathrm{d}\psi}^* = \mathrm{axial}(\underline{\underline{R}}^T\mathrm{d}\underline{\underline{R}})$ in the same bases, respectively. The incremental motion vector of point **A** is now defined as

$$\underline{\mathrm{d}\mathcal{U}}^* = \begin{Bmatrix} \underline{\underline{R}}^T\mathrm{d}\underline{u} \\ \underline{\mathrm{d}\psi}^* \end{Bmatrix}. \tag{42}$$

Consider now the following frame change operation

$$\underline{\mathrm{d}\mathcal{U}} = \begin{Bmatrix} \mathrm{d}\underline{u} + \widetilde{u}\underline{\mathrm{d}\psi} \\ \underline{\mathrm{d}\psi} \end{Bmatrix} = \underline{\underline{\mathcal{C}}}(\underline{u}, \underline{\underline{R}})\underline{\mathrm{d}\mathcal{U}}^*. \tag{43}$$

The change of frame operation involves two consecutive operations: a change of basis operation followed by a change of reference point operation. The first three components of the incremental motion vector, $\mathrm{d}\underline{u} + \widetilde{u}\,\underline{\mathrm{d}\psi}$, represent the components of the incremental displacement of the rigid body that instantaneously coincides with the origin of the reference frame, point **O**, resolved in basis $\mathcal{I}$. The next three components are the components of the incremental rotation vector resolved in the same basis.

4.4 Frame change for the applied load vector

Let $\underline{F}$ and $\underline{M}$ be the force and moment applied to the rigid body at point **A**, see fig 2. The components of these two vectors resolved in basis $\mathcal{B}^*$ form the load vector, $\underline{\mathcal{A}}^{*T} = \left\{\underline{F}^{*T}, \underline{M}^{*T}\right\}$. The following change of frame operation is now considered

$$\underline{\mathcal{A}} = \begin{Bmatrix} \underline{F} \\ \underline{M} + \widetilde{u}\,\underline{F} \end{Bmatrix} = \underline{\underline{\mathcal{C}}}^{-T}(\underline{u}, \underline{\underline{R}})\underline{\mathcal{A}}^*. \tag{44}$$

The change of frame operation involves two consecutive operations: a change of basis operation followed by a change of reference point operation. The first three components of the load vector are the components of the force vector resolved in basis $\mathcal{I}$. The next three components, $\underline{M} + \widetilde{u}\,\underline{F}$, represent the moment applied to the rigid body computed with respect to the point of the rigid body that instantaneously coincides with point **O**, resolved in basis $\mathcal{I}$. Loads $\underline{\mathcal{A}}$ and $\underline{\mathcal{A}}^*$ are equipollent, but their components are expressed in two different bases.

It is clear from the presentation of the previous sections that the linear and angular parts of motion are treated here in a coupled manner. For instance, the first three entries of the incremental motion "array" defined in eq. (43) store the components of the incremental displacement vector, and the next three entries store the components of the incremental rotation vector. Similarly, the first three entries of the load "array" defined in eq. (44) store the components of the force vector, and the next three entries store the components of the moment vector. In this paper, these arrays are called vectors because they comprise two vectors; the calligraphic notation used for these quantities is a reminder of their composite nature. Similarly, the motion tensor defined in eq. (26a) is comprised of four, second order tensors.

4.5 The velocity vector

Time derivatives of the rotation tensor give rise to the angular velocity vector given by eq. (5). The present section focuses on the study of time derivatives of the motion tensor, which leads to both velocity and angular velocity vectors. Expression $\widetilde{\omega} = \dot{\underline{\underline{R}}}\,\underline{\underline{R}}^T$, associated with the rotation tensor, defines the angular velocity vector. The corresponding expression, $\dot{\underline{\underline{\mathcal{C}}}}\,\underline{\underline{\mathcal{C}}}^{-1}$, associated with the motion tensor, leads to

$$\dot{\underline{\underline{\mathcal{C}}}}\,\underline{\underline{\mathcal{C}}}^{-1} = \begin{bmatrix} \widetilde{\omega} & (\widetilde{\dot{\underline{u}} + \widetilde{u}\underline{\omega}}) \\ \underline{\underline{0}} & \widetilde{\omega} \end{bmatrix} = \begin{bmatrix} \widetilde{\omega} & \widetilde{v} \\ \underline{\underline{0}} & \widetilde{\omega} \end{bmatrix}. \tag{45}$$

This expression gives rise to the velocity vector of the rigid body, $\underline{v} = \dot{\underline{u}} + \widetilde{u}\underline{\omega}$, which can be interpreted as the linear velocity of the point of the rigid body that instantaneously coincides with the origin of the reference frame, point **O**.

The velocity vector of the rigid body resolved in the inertial frame is defined as $\underline{\mathcal{V}}^T = \left\{\underline{v}^T, \underline{\omega}^T\right\}$, and the following notation is introduced

$$\underline{\underline{\mathcal{W}}}(\underline{\mathcal{V}}) = \begin{bmatrix} \widetilde{\omega} & \widetilde{v} \\ \underline{\underline{0}} & \widetilde{\omega} \end{bmatrix}. \tag{46}$$

Expression $\widetilde{\omega}^* = \underline{\underline{R}}^T\dot{\underline{\underline{R}}}$, associated with the rotation tensor, defines the components of the angular velocity vector in the rotating frame. The corresponding expression, $\underline{\underline{\mathcal{C}}}^{-1}\dot{\underline{\underline{\mathcal{C}}}}$, associated with the motion tensor, leads to

$$\underline{\underline{\mathcal{C}}}^{-1}\dot{\underline{\underline{\mathcal{C}}}} = \begin{bmatrix} \widetilde{\omega}^* & \widetilde{\underline{\underline{R}}^T\dot{\underline{u}}} \\ \underline{\underline{0}} & \widetilde{\omega} \end{bmatrix} = \begin{bmatrix} \widetilde{\omega}^* & \widetilde{v}^* \\ \underline{\underline{0}} & \widetilde{\omega}^* \end{bmatrix}. \tag{47}$$

This expression gives rise to two quantities. First, the components of the angular velocity of the rigid body resolved in the rotating basis, $\underline{\omega}^* = \text{axial}(\underline{\underline{R}}^T\dot{\underline{\underline{R}}})$. Second, the components of the velocity vector of the reference point of rigid body resolved in the rotating basis, $\underline{v}^* = \underline{\underline{R}}^T\dot{\underline{u}}$. The components of the velocity vector of the rigid body resolved in the material frame are now defined as

$$\underline{\mathcal{V}}^* = \begin{Bmatrix} \underline{v}^* \\ \underline{\omega}^* \end{Bmatrix} = \underline{\underline{\mathcal{C}}}^{-1}\underline{\mathcal{V}}. \tag{48}$$

The above developments are summarized in the following relationships

$$\dot{\underline{\underline{\mathcal{C}}}}\,\underline{\underline{\mathcal{C}}}^{-1} = \underline{\underline{\mathcal{W}}}(\underline{\mathcal{V}}), \tag{49a}$$

$$\underline{\underline{\mathcal{C}}}^{-1}\dot{\underline{\underline{\mathcal{C}}}} = \underline{\underline{\mathcal{W}}}(\underline{\mathcal{V}}^*). \tag{49b}$$

5 The vectorial parameterization of motion

The *vectorial parameterization of motion* consists of a minimal set of parameters defining the components of a *motion parameter vector*, which itself, consists of the components of two vectors. Bauchau and Choi [6] introduced this parameterization by selecting the motion parameter vector to be parallel to the eigenvector of the motion tensor associated with its unit eigenvalue. While this parallelism is an important property of the vectorial parameterization of motion, its tensorial nature is equally important and is explored in the following sections.

5.1 Fundamental properties

Consider three motions characterized by displacement vectors, $\underline{u}_1$, $\underline{u}_2$, and $\underline{u}_3$, and rotation tensors, $\underline{\underline{R}}_1$, $\underline{\underline{R}}_2$, and $\underline{\underline{R}}_3$, respectively. The three motions, denoted $(\underline{u}_1, \underline{\underline{R}}_1)$, $(\underline{u}_2, \underline{\underline{R}}_2)$, and $(\underline{u}_3, \underline{\underline{R}}_3)$, respectively are associated with three motion tensors, $\underline{\underline{\mathcal{C}}}_1$, $\underline{\underline{\mathcal{C}}}_2$, and $\underline{\underline{\mathcal{C}}}_3$, respectively, through the intrinsic expression for the motion tensor, eq. (39a).

Assume that the following triple product of motion tensors relates these three quantities,

$$\underline{\underline{\mathcal{C}}}_3 = \underline{\underline{\mathcal{C}}}_2^{-1}\underline{\underline{\mathcal{C}}}_1\underline{\underline{\mathcal{C}}}_2. \tag{50}$$

This operation corresponds to a change of frame operation for motion tensors: $\underline{\underline{\mathcal{C}}}_1$ and $\underline{\underline{\mathcal{C}}}_3$ are the components of the same motion tensor expressed in two frames related by motion tensor $\underline{\underline{\mathcal{C}}}_2$.

Using the intrinsic expression for the motion tensor, eq. (39a), eq. (50) now becomes

$$\begin{aligned}\underline{\underline{\mathcal{C}}}_3 &= \underline{\underline{\mathcal{I}}} + \underline{\underline{\mathcal{Z}}}(d_1c_1, \sin\phi_1)\underline{\underline{\mathcal{W}}}(\underline{\underline{\mathcal{C}}}_2^{-1}\underline{\mathcal{N}}_1) \\ &\quad + \underline{\underline{\mathcal{Z}}}(d_1c_2, 1-\cos\phi_1)\underline{\underline{\mathcal{W}}}^2(\underline{\underline{\mathcal{C}}}_2^{-1}\underline{\mathcal{N}}_1),\end{aligned}$$

where eq. (35) was used. Comparing this result with the intrinsic expression for the motion tensor implies that

$$\phi_3 = \phi_1, \tag{51a}$$

$$\begin{Bmatrix} \underline{m}_3 \\ \bar{n}_3 \end{Bmatrix} = \underline{\mathcal{N}}_3 = \underline{\underline{\mathcal{C}}}_2^{-1} \underline{\mathcal{N}}_1 = \begin{Bmatrix} \underline{\underline{R}}_2^T (\underline{m}_1 + \widetilde{n}_1 \underline{u}_2) \\ \underline{\underline{R}}_2^T \bar{n}_1 \end{Bmatrix}. \tag{51b}$$

These equations express the two conditions required for the proper transformation of motion tensors components under a change of frame. Note that an additional condition is required, $d_3 = d_1$, but is implied by eqs. (51). Indeed, eq. (51b) yields $\bar{n}_3^T \underline{m}_3 = \bar{n}_1^T \underline{\underline{R}}_2 \underline{\underline{R}}_2^T (\underline{m}_1 + \widetilde{n}_1 \underline{u}_2) = \bar{n}_1^T \underline{m}_1$, or $\lambda_3 = \lambda_1$, where λ is defined by eq. (28). In view of eq. (51a) and (28), $\lambda_3 = \lambda_1$ then yields $d_3 = d_1$.

Let $p(\phi)$ be an arbitrary scalar function of angle ϕ; eq. (51a) then implies $p(\phi_3) = p(\phi_1)$. Multiplication of eq. (51b) by $p_3 = p(\phi_3)$ on the left-hand side and $p_1 = p(\phi_1) = p(\phi_3)$ on the right-hand side then yields

$$\begin{Bmatrix} p_3 \underline{m}_3 \\ p_3 \bar{n}_3 \end{Bmatrix} = \begin{Bmatrix} \underline{\underline{R}}_2^T p_1 (\underline{m}_1 + \widetilde{n}_1 \underline{u}_2) \\ \underline{\underline{R}}_2^T p_1 \bar{n}_1 \end{Bmatrix}. \tag{52}$$

This equation is equivalent to eqs. (51). Indeed, the last three equations of eqs. (51b) and (52) are easily shown to be identical though a reasoning identical to that used to show the equivalence of eqs. (7) and (8).

The *vectorial parametrization of motion* is defined as

$$\underline{\mathcal{P}} = p(\phi) \underline{\mathcal{N}}, \tag{53}$$

where $\underline{\mathcal{P}}$ is the motion parameter vector. Equation (52) can now be recast in a more compact manner as

$$\underline{\mathcal{P}}_3 = \underline{\underline{\mathcal{C}}}_2^{-1} \underline{\mathcal{P}}_1. \tag{54}$$

The discussion presented above establishes that the tensorial nature of the motion tensor expressed by the transformation rule of its components, eq. (50), implies the tensorial nature of the rotation parameter vector expressed by the transformation rule of its components, eq. (54). It is easily shown that the process can be reversed, *i.e.*, tensorial nature of the rotation parameter vector implies that of the rotation tensor.

In summary, the vectorial parameterization of motion presents two fundamental properties.

(1) The vectorial parameterization of motion is tensorial in nature, as expressed by the following equivalence,

$$\underline{\underline{\mathcal{C}}}(\underline{\mathcal{P}}_3) = \underline{\underline{\mathcal{C}}}_2^{-1} \underline{\underline{\mathcal{C}}}(\underline{\mathcal{P}}_1) \underline{\underline{\mathcal{C}}}_2 \Leftrightarrow \underline{\mathcal{P}}_3 = \underline{\underline{\mathcal{C}}}_2^{-1} \underline{\mathcal{P}}_1. \tag{55}$$

The tensorial nature of the second-order motion tensor implies and is implied by the tensorial nature of the motion parameter vector, a first-order tensor.

(2) Motion parameter vectors are parallel to an *eigenvector of the motion tensor* associated with its unit eigenvalue. Equation (41) shows that vector $\underline{\mathcal{N}}$ is a linear combination of two linearly independent eigenvectors of the motion tensor, both associated with its unit eigenvalue; equation (53) then implies that the motion parameter vector shares this property.

Because these two properties imply each other, either can be taken as the definition of the vectorial parameterization of motion. A parameterization of motion is vectorial if and only if the motion parameter vector is parallel an eigenvector of the motion tensor associated with its unit eigenvalue.

5.2 The motion parameter vector

A more explicit expression of the motion parameter vector is as follows

$$\underline{\mathcal{P}} = \begin{Bmatrix} \underline{q} \\ \underline{p} \end{Bmatrix} = p \underline{\mathcal{N}} = \begin{Bmatrix} p \underline{m} \\ p \bar{n} \end{Bmatrix} = \begin{Bmatrix} p \underline{\underline{E}}(\phi) \underline{u} \\ p \bar{n} \end{Bmatrix} = \begin{Bmatrix} \underline{\underline{D}}(\underline{p}) \underline{u} \\ \underline{p} \end{Bmatrix}, \tag{56}$$

where $\underline{p}$ is the vectorial parameterization of rotation and tensor $\underline{\underline{E}}$ is defined by eq. (31). Using the notation developed for the vectorial parameterization of rotation, tensor $\underline{\underline{D}}$ becomes

$$\underline{\underline{D}}(\underline{p}) = \delta_0 \underline{\underline{I}} - \frac{1}{2} \widetilde{p} + \delta_2 \widetilde{p}\widetilde{p}, \tag{57}$$

where functions $\delta_0(\phi)$ and $\delta_2(\phi)$ are given by

$$\delta_0 = \frac{\alpha}{\nu}, \tag{58a}$$

$$\delta_2 = \frac{1}{p^2} \left(\delta_0 - \frac{1}{\varepsilon} \right). \tag{58b}$$

The determinant of this tensor is $\det(\underline{\underline{D}}) = \alpha/\nu^3$.

The inverse of this tensor is readily found as

$$\underline{\underline{F}}(\underline{p}) = \underline{\underline{D}}^{-1}(\underline{p}) = \varphi_0 \underline{\underline{I}} + \varphi_1 \widetilde{p} + \varphi_2 \widetilde{p}\widetilde{p}, \tag{59}$$

where functions $\varphi_0(\phi)$, $\varphi_1(\phi)$, and $\varphi_2(\phi)$ are given by

$$\varphi_0 = \frac{\nu}{\alpha}, \tag{60a}$$

$$\varphi_1 = \zeta_2, \tag{60b}$$

$$\varphi_2 = \frac{1}{p^2} (\varphi_0 - \zeta_1). \tag{60c}$$

where coefficients ζ_1 and ζ_2 are given by eqs. (14).

Tensor $\underline{\underline{F}}$ enjoys the following remarkable properties

$$\underline{\underline{R}} = \underline{\underline{F}}\, \underline{\underline{F}}^{-T} = \underline{\underline{F}}^{-T} \underline{\underline{F}}, \tag{61a}$$

$$\underline{\underline{R}} - \underline{\underline{I}} = \underline{\underline{F}}\, \widetilde{p} = \widetilde{p} \underline{\underline{F}}, \tag{61b}$$

$$\widetilde{p} = \underline{\underline{F}}^{-T} - \underline{\underline{F}}^{-1}, \tag{61c}$$

which are similar to those of the tangent tensor, eqs. (24).

The motion parameter vector is not yet fully defined yet because it depends on the choice of the generating function, $p(\phi)$, of the vectorial parameterization of rotation and furthermore, scalar function $\alpha(\phi)$ can be selected arbitrarily. Generating functions must be odd functions of the rotation angle and present the limit behavior expressed by eq. (12), *i.e.*, all rotation parameter vectors must approach the infinitesimal rotation vector when $\phi \to 0$.

Similarly, the displacement related part of the motion parameter vector, $\underline{q}$, should approach the infinitesimal displacement vector when $\underline{u} \to 0$ and $\phi \to 0$. In view of eq. (56), this requirement implies $\lim_{\phi\to 0} \underline{\underline{D}}(\underline{p}) = \underline{\underline{I}}$, or $\lim_{\phi\to 0} \alpha/\nu = 1$, and finally

$$\lim_{\phi\to 0} \alpha = 1. \tag{62}$$

5.2.1 Time derivative of the displacement

In the manipulation of the time derivatives of the motion tensor, it will be necessary to evaluate $\underline{\dot{u}} = (\underline{\underline{F}}\,\underline{q})^{\cdot} = \underline{\underline{\dot{F}}}\,\underline{q} + \underline{\underline{F}}\,\underline{\dot{q}}$, which can be written as $\underline{\dot{u}} = \underline{\underline{L}}(\underline{q},\underline{p})\underline{\dot{p}} + \underline{\underline{F}}\,\underline{\dot{q}}$, where operator $\underline{\underline{L}}$ is implicitly defined as follows

$$\underline{\underline{\dot{F}}}(\underline{p})\underline{q} = \underline{\underline{L}}(\underline{q},\underline{p})\underline{\dot{p}}. \tag{63}$$

Using eq. (59), operator $\underline{\underline{L}}$ is easily found as

$$\underline{\underline{L}}(\underline{q},\underline{p}) = \frac{1}{p'}\left(\frac{\varphi_0'}{p} + \frac{\varphi_1'}{p}\widetilde{p} + \frac{\varphi_2'}{p}\widetilde{p}\,\widetilde{p}\right)\underline{q}\,\underline{p}^T \\ - \varphi_1\widetilde{q} - \varphi_2\left(2\widetilde{p}\,\widetilde{q} - \widetilde{q}\,\widetilde{p}\right).$$

Operator $\underline{\underline{L}}$ enjoys the following properties,

$$\underline{\underline{R}}_1^T\underline{\underline{L}}(\underline{q},\underline{p})\underline{\underline{R}}_1 = \underline{\underline{L}}(\underline{\underline{R}}_1^T\underline{q},\underline{\underline{R}}_1^T\underline{p}), \tag{64a}$$

$$\underline{\underline{L}}(\underline{q}_1+\underline{q}_2,\underline{p}) = \underline{\underline{L}}(\underline{q}_1,\underline{p}) + \underline{\underline{L}}(\underline{q}_2,\underline{p}), \tag{64b}$$

$$\underline{\underline{L}}\,\widetilde{p} = \widetilde{\underline{\underline{F}}\,\underline{q}} - \underline{\underline{F}}\,\widetilde{q}, \tag{64c}$$

$$\widetilde{p}\,\underline{\underline{L}} = \widetilde{\underline{\underline{\dot{F}}}\,\underline{q}} - \underline{\underline{R}}\,\widetilde{q}\,\underline{\underline{H}}^T, \tag{64d}$$

$$\underline{\underline{L}}(\widetilde{p}\,\underline{q},\underline{p}) = \widetilde{p}\,\underline{\underline{L}}(\underline{q},\underline{p}) - \underline{\underline{L}}(\underline{q},\underline{p})\,\widetilde{p}. \tag{64e}$$

The first property, eq. (64a), expresses the transformation of the components of operator $\underline{\underline{L}}$ under a change of basis of both of its arguments. The second property, eq. (64b), expresses the linearity of operator $\underline{\underline{L}}$ with respect to its first argument. Property (64d) stems from the definition of operator $\underline{\underline{L}}$, $\widetilde{p}\underline{\underline{\dot{F}}}\,\underline{q} = \widetilde{p}\underline{\underline{L}}\,\underline{\dot{p}}$, and noting that eq. (61b) implies $\widetilde{p}\underline{\underline{\dot{F}}} = \underline{\underline{\dot{R}}} - \dot{\widetilde{p}}\underline{\underline{F}}$.

5.3 The generalized vector product tensor

The skew-symmetric tensor $\widetilde{p}$ plays an important role in the vectorial parameterization of rotation as it appears in the explicit expression of all rotation related tensors. The generalized vector product tensor, $\underline{\underline{\mathcal{W}}}$, defined by eq. (32), can also be written for the motion parameter vector as

$$\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}) = \begin{bmatrix} \widetilde{p} & \widetilde{q} \\ \underline{\underline{0}} & \widetilde{p} \end{bmatrix}. \tag{65}$$

This operator plays an important role in the vectorial parameterization of motion.

The tensorial nature of the generalized vector product operator directly follows from eq. (35), leading to

$$\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}_3) = \underline{\underline{\mathcal{C}}}_2^{-1}\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}_1)\underline{\underline{\mathcal{C}}}_2 \Leftrightarrow \underline{\mathcal{P}}_3 = \underline{\underline{\mathcal{C}}}_2^{-1}\underline{\mathcal{P}}_1. \tag{66}$$

This statement generalizes eq. (4), which expresses the tensorial nature of the skew-symmetric operator, $\widetilde{p}$.

Identity (36) generalizes as

$$\underline{\underline{\mathcal{W}}}^3(\underline{\mathcal{P}}) + \underline{\underline{\mathcal{Z}}}(2\varrho, p^2)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}) = 0, \tag{67}$$

where scalar ϱ is closely related to the intrinsic displacement of the rigid body,

$$\varrho = \underline{p}^T\underline{q} = \frac{pd}{\varphi_0}. \tag{68}$$

5.4 Basic operators

The motion tensor and its inverse are obtained from eqs. (39) as

$$\underline{\underline{\mathcal{C}}}(\underline{\mathcal{P}}) = \underline{\underline{\mathcal{I}}} + \underline{\underline{\mathcal{Z}}}(\bar{\zeta}_1,\zeta_1)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}) + \underline{\underline{\mathcal{Z}}}(\bar{\zeta}_2,\zeta_2)\underline{\underline{\mathcal{W}}}^2(\underline{\mathcal{P}}), \tag{69a}$$

$$\underline{\underline{\mathcal{C}}}^{-1}(\underline{\mathcal{P}}) = \underline{\underline{\mathcal{I}}} - \underline{\underline{\mathcal{Z}}}(\bar{\zeta}_1,\zeta_1)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}) + \underline{\underline{\mathcal{Z}}}(\bar{\zeta}_2,\zeta_2)\underline{\underline{\mathcal{W}}}^2(\underline{\mathcal{P}}). \tag{69b}$$

The parallel between the expressions for the rotation and motion tensors, eqs. (13) and (69), is now evident. Coefficients ζ_1 and ζ_2 are given by eqs. (14), and

$$\bar{\zeta}_1 = \varrho(\varphi_2 - \varphi_0\zeta_2), \tag{70a}$$

$$\bar{\zeta}_2 = \varrho(\varphi_2\zeta_1 - \zeta_2^2). \tag{70b}$$

The following multiplicative decomposition of the motion tensor generalizes the corresponding expression for the rotation tensor, eq. (17),

$$\underline{\underline{\mathcal{C}}}(\underline{\mathcal{P}}) = \left[\underline{\underline{\mathcal{I}}} + \frac{1}{2}\underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon)\underline{\underline{\mathcal{W}}}\right]\left[\underline{\underline{\mathcal{I}}} - \frac{1}{2}\underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon)\underline{\underline{\mathcal{W}}}\right]^{-1} \\ = \left[\underline{\underline{\mathcal{I}}} - \frac{1}{2}\underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon)\underline{\underline{\mathcal{W}}}\right]^{-1}\left[\underline{\underline{\mathcal{I}}} + \frac{1}{2}\underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon)\underline{\underline{\mathcal{W}}}\right]. \tag{71}$$

Furthermore, eq. (18) also generalizes as

$$\left[\underline{\underline{\mathcal{I}}} - \frac{1}{2}\underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon)\underline{\underline{\mathcal{W}}}\right]^{-1} = \frac{\underline{\underline{\mathcal{C}}} + \underline{\underline{\mathcal{I}}}}{2}, \tag{72a}$$

$$\left[\underline{\underline{\mathcal{I}}} + \frac{1}{2}\underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon)\underline{\underline{\mathcal{W}}}\right]^{-1} = \frac{\underline{\underline{\mathcal{C}}}^{-1} + \underline{\underline{\mathcal{I}}}}{2}. \tag{72b}$$

In these last two equations, coefficient ε is given by eq. (15b) and

$$\zeta_1\bar{\varepsilon} = 2\bar{\zeta}_2 - \varepsilon\bar{\zeta}_1. \tag{73}$$

The velocity vector is obtained from a time derivative of the motion tensor, as indicated in eq. (45). For the vectorial parameterization of motion, this becomes

$$\underline{\mathcal{V}} = \begin{Bmatrix} \underline{v} \\ \underline{\omega} \end{Bmatrix} = \begin{Bmatrix} \dot{\underline{\underline{F}}}(\underline{p})\underline{q} + \underline{\underline{F}}(\underline{p})\dot{\underline{q}} + \widetilde{\underline{\underline{F}}(\underline{p})\underline{q}}\ \underline{\underline{H}}(\underline{p})\dot{\underline{p}} \\ \underline{\underline{H}}(\underline{p})\dot{\underline{p}} \end{Bmatrix}.$$

The velocity vector is now related to the time derivative of the motion parameter vector, $\underline{\mathcal{V}} = \underline{\underline{\mathcal{H}}}\dot{\underline{\mathcal{P}}}$, where tensor $\underline{\underline{\mathcal{H}}}$ is defined by eq. (75a). Using eq. (47), similar developments for the components of the velocity vector resolved in the material frame lead to $\underline{\mathcal{V}}^* = \underline{\underline{\mathcal{H}}}^*\dot{\underline{\mathcal{P}}}$, where tensor $\underline{\underline{\mathcal{H}}}^*$ is defined by eq. (75b).

In summary, the components of the velocity vector resolved in the inertial and material frames, denoted $\underline{\mathcal{V}}$ and $\underline{\mathcal{V}}^*$, respectively, are related to the time derivatives of the motion parameter vectors through the following relationships,

$$\underline{\mathcal{V}} = \underline{\underline{\mathcal{H}}}\dot{\underline{\mathcal{P}}}, \tag{74a}$$

$$\underline{\mathcal{V}}^* = \underline{\underline{\mathcal{H}}}^*\dot{\underline{\mathcal{P}}}. \tag{74b}$$

Explicit expressions for tensors $\underline{\underline{\mathcal{H}}}$, $\underline{\underline{\mathcal{H}}}^*$, and their inverses are

$$\underline{\underline{\mathcal{H}}} = \begin{bmatrix} \underline{\underline{F}} & \underline{\underline{L}} + \widetilde{\underline{\underline{F}}\underline{q}}\ \underline{\underline{H}} \\ \underline{\underline{0}} & \underline{\underline{H}} \end{bmatrix}, \tag{75a}$$

$$\underline{\underline{\mathcal{H}}}^* = \begin{bmatrix} \underline{\underline{F}}^T & \underline{\underline{R}}^T\underline{\underline{L}} \\ \underline{\underline{0}} & \underline{\underline{H}}^T \end{bmatrix}, \tag{75b}$$

$$\underline{\underline{\mathcal{H}}}^{-1} = \begin{bmatrix} \underline{\underline{F}}^{-1} & -\underline{\underline{F}}^{-1}\left(\underline{\underline{L}}\,\underline{\underline{H}}^{-1} + \widetilde{\underline{\underline{F}}\,\underline{q}}\right) \\ \underline{\underline{0}} & \underline{\underline{H}}^{-1} \end{bmatrix}, \tag{75c}$$

$$\underline{\underline{\mathcal{H}}}^{*-1} = \begin{bmatrix} \underline{\underline{F}}^{-T} & -\underline{\underline{F}}^{-1}\underline{\underline{L}}\,\underline{\underline{H}}^{-T} \\ \underline{\underline{0}} & \underline{\underline{H}}^{-T} \end{bmatrix}, \tag{75d}$$

where operator $\underline{\underline{L}}(\underline{q},\underline{p})$ is defined by eq. (63).

5.4.1 Properties of tensor $\underline{\underline{H}}$

Tensors $\underline{\underline{\mathcal{H}}}$, $\underline{\underline{\mathcal{H}}}^*$, and their inverses enjoys the following remarkable properties,

$$\underline{\underline{\mathcal{C}}} = \underline{\underline{\mathcal{H}}}\,\underline{\underline{\mathcal{H}}}^{*-1}, \tag{76a}$$

$$\underline{\underline{\mathcal{C}}}^{-1} = \underline{\underline{\mathcal{H}}}^*\underline{\underline{\mathcal{H}}}^{-1}, \tag{76b}$$

$$\underline{\underline{\mathcal{C}}} - \underline{\underline{\mathcal{I}}} = \underline{\underline{\mathcal{W}}}\,\underline{\underline{\mathcal{H}}} = \underline{\underline{\mathcal{H}}}\,\underline{\underline{\mathcal{W}}}, \tag{76c}$$

$$\underline{\underline{\mathcal{C}}}^{-1} - \underline{\underline{\mathcal{I}}} = -\underline{\underline{\mathcal{W}}}\,\underline{\underline{\mathcal{H}}}^* = -\underline{\underline{\mathcal{H}}}^*\,\underline{\underline{\mathcal{W}}}, \tag{76d}$$

$$\underline{\underline{\mathcal{W}}} = \underline{\underline{\mathcal{H}}}^{*-1} - \underline{\underline{\mathcal{H}}}^{-1}. \tag{76e}$$

These properties are established directly from the definition of the tangent tensor, eqs. (75), taking into account the properties of the tangent tensor for the vectorial parameterization of rotation, eqs. (24), and those of tensor $\underline{\underline{F}}$, eqs. (61). The properties of operator $\underline{\underline{L}}$, eqs. (64), must also be used.

Operator $\underline{\underline{\mathcal{H}}}$ is specific to a particular vectorial parameterization, *i.e.*, its expression depends on the choice of the generating function. It is, however, a second-order tensor because the following equivalence holds

$$\underline{\underline{\mathcal{H}}}(\underline{\mathcal{P}}_3) = \underline{\underline{\mathcal{C}}}_2^{-1}\underline{\underline{\mathcal{H}}}(\underline{\mathcal{P}}_1)\underline{\underline{\mathcal{C}}}_2 \Leftrightarrow \underline{\mathcal{P}}_3 = \underline{\underline{\mathcal{C}}}_2^{-1}\underline{\mathcal{P}}_1. \tag{77}$$

Although tensor $\underline{\underline{\mathcal{H}}}$ is not an intrinsic tensor because it depends on the choice of a specific generating function, it is a second-order tensor for all vectorial parameterizations of motion. Equation (77) is established directly from the definition of tensor $\underline{\underline{\mathcal{H}}}$, eq. (75a) by using eqs. (24), (61), and eqs. (64).

5.5 Selection of function α

The vectorial parameterization of motion presented in the previous sections consists of a set of displacement related parameters, $\underline{q} = \underline{\underline{D}}\,\underline{u}$, and of the rotation parameter vector, $\underline{p}$. The motion parameter vector is parallel to an eigenvector of the motion tensor associated with its unit eigenvalue. This leads to families of parameterizations that depend on two choices: the choice of the generating function, $p(\phi)$, and that of the arbitrary function, $\alpha(\phi)$, appearing in tensor $\underline{\underline{D}}$.

As discussed in ref. [5], the generating function can be selected to simplify some of the operators involved in manipulating rotations. But more importantly, judicious choices of this function can eliminate singularities that occur in the various rotation operators.

The occurrence of singularities is also a major concern when dealing with the vectorial parameterization of motion. Two criteria guide the selection of function $\alpha(\phi)$. First, a one to one, singularity free relationship must exist between the displacement related part of motion parameter vector, $\underline{q}$, and the physical displacement vector, $\underline{u}$. Second, the limit behavior expressed by eq. (62) must be satisfied.

5.5.1 Alternative choices of the motion parameter vector

The presence of the arbitrary function, $\alpha(\phi)$, reflects the non-uniqueness of the eigenvector of the motion tensor associated with its unit eigenvalue. An additional constraint is required to evaluate this function. For instance, imposing the orthogonality condition, $\underline{q}^T\underline{p} = 0$, leads to $\alpha = 0$, and the resulting motion parameter vector then corresponds to the Plücker coordinates of the Mozzi-Chasles axis. This choice, however, does satisfy the limit behavior condition, eq. (62), and furthermore, because $\det(\underline{\underline{D}}) = \alpha/\nu^3 = 0$, a one-to-one mapping between $\underline{q}$ and $\underline{u}$ ceases to exist.

A simple choice is to select $\alpha = 1$ it then follows that $\underline{\underline{D}} = \underline{\underline{G}}^T/\nu$ and $\underline{\underline{F}} = \nu\underline{\underline{G}}$. The determinant of tensor $\underline{\underline{D}}$ is $\det(\underline{\underline{D}}) = 1/\nu^3$. This choice satisfies the limit behavior condition, eq. (62).

The close connection between tensors $\underline{\underline{D}}$ and $\underline{\underline{H}}^{-1}$ is evident when comparing eqs. (57) and (22). Selecting $\alpha = \nu p'$ results in $\underline{\underline{D}} = \underline{\underline{H}}^{-1}$ and $\underline{\underline{F}} = \underline{\underline{H}}$. The determinant of tensor $\underline{\underline{D}}$ is $\det(\underline{\underline{D}}) = p'/\nu^2$. Here again the limit behavior condition is satisfied.

It is possible to eliminate the quadratic term in $\underline{p}$ of tensor $\underline{\underline{D}}$ by choosing $\alpha = \nu/\varepsilon$. This leads to the following expressions: $\underline{\underline{D}} = (\underline{\underline{I}} - \varepsilon\widetilde{p}/2)/\varepsilon$ and $\underline{\underline{F}} = \varepsilon(\underline{\underline{R}} + \underline{\underline{I}})/2$. The determinant of tensor $\underline{\underline{D}}$ is $\det(\underline{\underline{D}}) = 1/(\varepsilon\nu^2)$.

One final alternative is to select $\alpha = \varepsilon/\nu$, leading to $\underline{\underline{D}} = (\underline{\underline{I}} + \underline{\underline{R}}^T)/(2\zeta_1)$, $\underline{\underline{F}} = \zeta_1(\underline{\underline{I}} + \varepsilon\widetilde{p}/2)$, and $\det(\underline{\underline{D}}) = 1/(\nu^2\zeta_1)$. This choice leads to Cayley's motion parameters [13].

Of all the choices presented in this section, $\alpha = \nu p'$ seems to the most desirable because it leads to $\underline{\underline{D}} = \underline{\underline{H}}^{-1}$. This choice satisfies the limit behavior expressed by eq. (62), and because the tangent tensor, $\underline{\underline{H}}$, plays a critical role in manipulating rotations, eliminating singularities from this tensor is already a criterion for the selection of appropriate generating functions. A singularity free tangent tensor in the vectorial parameterization of rotation will then automatically lead to a one-to-one mapping between $\underline{q}$ and $\underline{u}$, avoiding the occurrence of singularities in the vectorial parameterization of motion.

5.5.2 Alternative expressions of tangent tensor

Tangent tensors $\underline{\underline{\mathcal{H}}}$, $\underline{\underline{\mathcal{H}}}^*$, and their inverses are given by eqs. (75), which are valid for any choice of parameter α. If this parameter is selected to be $\alpha = \nu p'$, alternative expressions of tangent tensor can be obtained,

$$\begin{aligned}\underline{\underline{\mathcal{H}}}(\underline{\mathcal{P}}) &= \underline{\underline{\mathcal{Z}}}(\bar{\sigma}_0, \sigma_0) + \underline{\underline{\mathcal{Z}}}(\bar{\zeta}_2, \zeta_2)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}) \\ &\quad + \underline{\underline{\mathcal{Z}}}(\bar{\sigma}_2, \sigma_2)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}})\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}), \qquad (78a)\\ \underline{\underline{\mathcal{H}}}^*(\underline{\mathcal{P}}) &= \underline{\underline{\mathcal{Z}}}(\bar{\sigma}_0, \sigma_0) - \underline{\underline{\mathcal{Z}}}(\bar{\zeta}_2, \zeta_2)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}) \\ &\quad + \underline{\underline{\mathcal{Z}}}(\bar{\sigma}_2, \sigma_2)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}})\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}), \qquad (78b)\\ \underline{\underline{\mathcal{H}}}^{-1}(\underline{\mathcal{P}}) &= \underline{\underline{\mathcal{Z}}}(\bar{\chi}_0, \chi_0) - \tfrac{1}{2}\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}) \\ &\quad + \underline{\underline{\mathcal{Z}}}(\bar{\chi}_2, \chi_2)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}})\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}), \qquad (78c)\\ \underline{\underline{\mathcal{H}}}^{*-1}(\underline{\mathcal{P}}) &= \underline{\underline{\mathcal{Z}}}(\bar{\chi}_0, \chi_0) + \tfrac{1}{2}\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}) \\ &\quad + \underline{\underline{\mathcal{Z}}}(\bar{\chi}_2, \chi_2)\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}})\underline{\underline{\mathcal{W}}}(\underline{\mathcal{P}}). \qquad (78d)\end{aligned}$$

The parallel between these expressions and those for the vectorial parameterization of rotation, eqs. (20) and (22), is evident.

Coefficients σ_0, σ_1, and σ_2 are given by eqs. (21), and

$$\bar{\sigma}_0 = \varrho\varphi_0'/(pp'), \qquad (79a)$$

$$p^2\bar{\sigma}_2 = \bar{\sigma}_0 - 2\varrho\varphi_2 - \bar{\zeta}_1. \qquad (79b)$$

Coefficient χ_0 and χ_2 are given by eqs. (23), and

$$\bar{\chi}_0 = -\chi_0\delta_0\bar{\sigma}_0, \qquad (80a)$$

$$p^2\zeta_2\bar{\chi}_2 = \frac{\bar{\zeta}_2}{\varepsilon} - \frac{\bar{\zeta}_1}{2} + \bar{\chi}_0\zeta_2 - 2\varrho\zeta_2\delta_2. \qquad (80b)$$

6 Conclusions

Tensor analysis expresses the invariance of the laws of physics with respect to change of basis and change of frame operations. Mechanics formulations involving finite rotation and motion should be based parameterizations that give rise to tensor quantities. This paper has presented two formal proofs. First, rotation parameter vectors are tensors if and only if they are parallel to the eigenvector of the rotation tensor associated with its unit eigenvalue. Second, motion parameter vectors are tensors if and only if they are parallel to the eigenvector of the motion tensor associated with its unit eigenvalue. Under these conditions, families of tensorial parameterizations of rotation and motion are obtained for which generic expressions of the rotation, motion, and tangent tensors can be obtained. Because they are based on a unified formulation, the expressions for the rotation and motion related tensors are very similar to each other, and furthermore, tensor identities associated with rotation have their counterpart associated with motion. Applications of the formulation developed in this paper will be presented in an upcoming paper dealing tensorial deformation measures for finite elastic bodies.

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