Tensorial Deformation Measures for Flexible Joints*

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Abstract

Flexible joints, sometimes called bushing elements or force elements, are found in all multibody dynamics codes. In their simplest form, flexible joints simply consist of sets of three linear and three torsional springs placed between two nodes of a multibody system. For infinitesimal deformations, the selection of the lumped spring constants is an easy task, which can be based on a numerical simulation of the joint or on experimental measurements. If the joint undergoes finite deformations, identification of its stiffness characteristics is not so simple, specially if the joint is itself a complex system. When finite deformations occur, the definition of deformation measures becomes a critical issue. Indeed, for finite deformation, the observed nonlinear behavior of materials is partly due to material characteristics, and partly due to kinematics. This paper focuses on the determination of the proper finite deformation measures for elastic bodies of finite dimension. In contrast, classical strain measures, such as the Green-Lagrange or Almansi strains, among many others, characterize finite deformations of infinitesimal elements of a body. It is argued that proper finite deformation measures must be of a tensorial nature, i.e., *must present specific invariance characteristics. This requirement is satisfied if and only if deformation measures are parallel to the eigenvector of the motion tensor.*

1 Introduction

Flexible joints, sometimes called bushing elements or force elements, are found in all multibody dynamics codes. In their simplest form, flexible joints simply consist of sets of three linear and three

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torsional springs placed between two nodes of a multibody system. For infinitesimal deformations, the selection of the lumped spring constants is an easy task, which can be based on a numerical simulation of the joint or on experimental measurements.

If the joint undergoes finite deformations, identification of its stiffness characteristics is not so simple, specially if the joint is itself a complex system. When finite deformations occur, the definition of the deformation measures becomes a critical issue. Indeed, for finite deformation, the observed nonlinear behavior of materials is partly due to material characteristics, and partly due to kinematics.

For instance, Anand [1, 2] has shown that the classical strain energy function for infinitesimal isotropic elasticity is in good agreement with experiment for a wide class of materials for moderately large deformations, provided the infinitesimal strain measure used in the strain energy function is replaced by the Hencky or logarithmic measure of finite strain. This means that the behavior of materials for moderate deformations can be captured accurately using linear constitutive laws, but replacing the infinitesimal strain measures by finite deformation measures that are nonlinear functions of the displacements.

These nonlinear deformation measures capture the observed nonlinear behavior associated with the nonlinear kinematics of the problem. Degener *et al.* [3] also reported similar findings for the torsional behavior of beams subjected to large axial elongation.

Much attention has been devoted to the problem of synthesizing accurate constitutive properties for the modeling of flexible bushings presenting complex, time-dependent rheological behavior [4, 5]. It is worth stressing, however, that the literature seldom addresses three-dimensional joint deformations.

Much like multibody codes, most FE codes also support the modeling of lumped structural elements. While linear analysis is easily implemented, problems are encountered when dealing with finite displacements and rotations, as pointed out by Masarati and Morandini [6]. Structural analysis codes, either specifically intended for multibody dynamics analysis, like MSC/ADAMS, or for nonlinear FEA with multibody capabilities, like Abaqus/Standard, allow arbitrarily large absolute displacements and rotations of the nodes and correctly describe their rigid-body motion. When lumped deformable joints are used, relative displacements and rotations are often required to remain moderate, although not necessarily infinitesimal.

Such restrictions occur when using the FIELD element of MSC/ADAMS, a linear element that implements an orthotropic torsional spring based on a constant, orthotropic constitutive matrix [7]. Similarly, the JOINTC element implemented in Abaqus/Standard, describes the interaction between two nodes when the second node can "displace and rotate slightly with respect to the first node [8]," because its formulation is based on an approximate relative rotation measure.

The formulations and implementations of flexible joints available in research and commercial codes do not appear to allow arbitrarily large relative displacements and rotations. Moreover, in many cases, the ordering sequence of the nodes connected to the joint matters, because the behavior of the flexible joint is biased towards one of the nodes. This problem is known to experienced analysts using these codes. To the authors' knowledge, these facts are rarely acknowledged in the literature. It appears that little effort has been devoted to the elimination of these shortcomings from the formulations found in research and commercially available codes, although the predictions of these codes might be unexpected.

This paper presents families of finite deformation measures that can be used to characterize the deformation of flexible joints. These deformation measures are closely related to the tensorial parameterization motion developed by Bauchau and Li [9, 10]. Because they are of a tensorial nature, these deformation measures are intrinsic and invariant. Furthermore, it will be shown that using these strain measures in combination with the linear constitutive laws of the joint enable the accurate prediction of joint behavior under moderate deformation. Numerical examples, including the analysis of components that are widely used in the mechanical and aerospace industry, show

how the proposed formulation can easily and accurately account for nonlinear geometrical effects and thus deliver compact and accurate models suitable for the analysis of the global behavior of rather complex components. Section 2 describes the configuration of the flexible joint and the basic properties of the motion tensor are reviewed in section 3. Section 4 presents the proposed deformation measures, which are derived from invariance considerations.

2 Flexible joint configuration

Figure 1 shows a flexible joint in its reference and deformed configurations. It consists of a three-dimensional elastic body of finite dimension and of two rigid bodies, called handle k and handle ℓ, that are rigidly connected to the elastic body. In the reference configuration, the configuration of the handles is defined by frame $\mathcal{F}_0 = \left[\mathbf{K} = \mathbf{L}, \mathcal{B}_0 = (\bar{b}_{01}, \bar{b}_{02}, \bar{b}_{03})\right]$, where $\mathcal{B}_0$ forms an orthonormal basis. Points $\mathbf{K}$ and $\mathbf{L}$ are material points of handles k and ℓ, respectively, with coincident geometric locations.

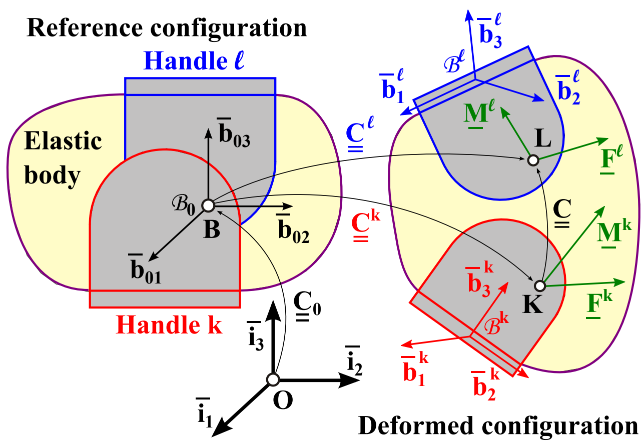

Figure 1: Configuration of the flexible joint

In the deformed configuration, the two handles move to new positions and the elastic body deforms. Points $\mathbf{K}$ and $\mathbf{L}$ are now at distinct locations; the relative displacement vector of point $\mathbf{L}$ with respect to point $\mathbf{K}$ is denoted $\underline{u}$. The configurations of the two handles are now distinct and two distinct frames, $\mathcal{F}^k = \left[\mathbf{K}, \mathcal{B}^k = (\bar{b}_1^k, \bar{b}_2^k, \bar{b}_3^k)\right]$ and $\mathcal{F}^\ell = \left[\mathbf{L}, \mathcal{B}^\ell = (\bar{b}_1^\ell, \bar{b}_2^\ell, \bar{b}_3^\ell)\right]$, define the configurations of handle k and ℓ, respectively. The relative rotation tensor of basis $\mathcal{B}^\ell$ with respect to basis $\mathcal{B}^k$ is denoted $\underline{\underline{R}}$.

The deformation of the flexible joint stems from applied forces and moments. At point $\mathbf{K}$, the applied force and moment vectors are denoted $\underline{F}_k$ and $\underline{M}_k$, respectively; the corresponding quantities applied at point $\mathbf{L}$ are denoted $\underline{F}_\ell$ and $\underline{M}_\ell$, respectively. The loading applied to the flexible joint is defined in the following manner

$$\underline{\mathcal{A}}_k = \begin{Bmatrix} \underline{F}_k \\ \underline{M}_k \end{Bmatrix}, \quad \underline{\mathcal{A}}_\ell = \begin{Bmatrix} \underline{F}_\ell \\ \underline{M}_\ell \end{Bmatrix}, \tag{1}$$

where $\underline{\mathcal{A}}_k$ and $\underline{\mathcal{A}}_\ell$ denote the loads applied at points $\mathbf{K}$ and $\mathbf{L}$, respectively. According to Newton's third law, these loads must be in equilibrium, *i.e.*,

$$\underline{\mathcal{A}}_k = -\begin{bmatrix} \underline{\underline{I}} & \underline{\underline{0}} \\ \tilde{u} & \underline{\underline{I}} \end{bmatrix} \underline{\mathcal{A}}_\ell, \tag{2}$$

where $\tilde{u}$ is the skew symmetric tensor associated with the components of vector $\underline{u}$, *i.e.*, $\underline{u} = \text{axial}(\tilde{u})$. The joint is assumed to be massless, *i.e.*, inertial forces associated with its motion are neglected.

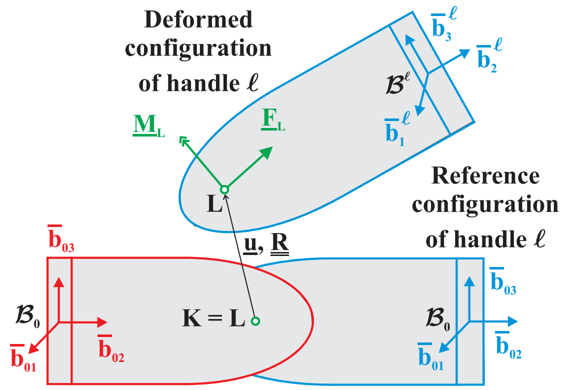

Figure 2: Configuration of the flexible joint for scenario ℓ. For clarity of the figure, the elastic body is not shown

The state of deformation of the elastic body depends on the relative displacement and rotation of the two handles and is unaffected by rigid body motions. Consequently, it is possible to assume that handle k does not move, and the relative displacement and rotation of handle ℓ with respect to handle k then simply becomes its absolute motion, as illustrated in fig. 2. This configuration is denoted scenario ℓ. Of course, scenario k could also be defined in a similar manner if the location of handle ℓ is assumed to remain fixed in space.

3 The motion tensor

The 6×6 *translation tensor* and its inverse are defined as

$$\underline{\underline{\mathcal{T}}}(\underline{u}) = \begin{bmatrix} \underline{\underline{I}} & \tilde{u} \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix}, \quad \underline{\underline{\mathcal{T}}}^{-1}(\underline{u}) = \begin{bmatrix} \underline{\underline{I}} & \tilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix}, \tag{3}$$

and the 6×6 *rotation tensor* and its inverse as

$$\underline{\underline{\mathcal{R}}} = \begin{bmatrix} \underline{\underline{R}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{R}} \end{bmatrix}, \quad \underline{\underline{\mathcal{R}}}^{-1} = \begin{bmatrix} \underline{\underline{R}}^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{R}}^T \end{bmatrix} = \underline{\underline{\mathcal{R}}}^T. \tag{4}$$

The *motion tensor* expresses the transformation rule for the Plücker coordinates of a material line of a rigid body under a frame change [10]. The motion tensor, $\underline{\underline{\mathcal{C}}}$, and its inverse are defined as

$$\underline{\underline{\mathcal{C}}}(\underline{u}, \underline{\underline{R}}) = \underline{\underline{\mathcal{T}}}\,\underline{\underline{\mathcal{R}}} = \begin{bmatrix} \underline{\underline{R}} & \tilde{u}\underline{\underline{R}} \\ 0 & \underline{\underline{R}} \end{bmatrix}, \tag{5a}$$

$$\underline{\underline{\mathcal{C}}}^{-1}(\underline{u}, \underline{\underline{R}}) = \underline{\underline{\mathcal{R}}}^{-1}\underline{\underline{\mathcal{T}}}^{-1} = \begin{bmatrix} \underline{\underline{R}}^T & \underline{\underline{R}}^T\tilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{R}}^T \end{bmatrix}. \tag{5b}$$

Tensor analysis expresses the invariance of the laws of physics with respect to a change of basis [11]. Consequently, the components of a vector change according to specific rules when resolved in different bases. For instance, if $\underline{F}_\ell^+$ and $\underline{F}_\ell^*$ denote the components of force vector $\underline{F}_\ell$ in bases $\mathcal{B}_0$ and $\mathcal{B}^\ell$, respectively, $\underline{F}_\ell^+ = \underline{\underline{R}}\,\underline{F}_\ell^*$, where $\underline{\underline{R}}$ denote the components of the rotation tensor that brings basis $\mathcal{B}^k$ to $\mathcal{B}^\ell$ resolved in basis $\mathcal{B}^k$. The following notational convention is introduced: the components of all vectors and tensors resolved in basis $\mathcal{B}^k$ will be denoted with a superscript $(\cdot)^+$ and the components of the same quantities resolved in basis $\mathcal{B}^\ell$ are denoted with a superscript $(\cdot)^*$.

The motion tensor generalizes these well-known concepts. Indeed, the laws of physics are invariant with respect to a change of basis, but also with respect to the change of origin. As expressed by eq. (5a), the motion tensor combines rotation and translation transformations into a single operation [12, 10].

The motion tensor plays an important role in this paper and its properties are summarized below.

3.1 Frame change for the differential motion vector

Consider the differential displacement of point **L** shown in fig 2. The components of this differential displacement vector in bases $\mathcal{B}^k$ and $\mathcal{B}^\ell$ are $\mathrm{d}\underline{u}^+$ and $\underline{\underline{R}}^T \mathrm{d}\underline{u}^+$, respectively. The components of the differential rotation vector of handle ℓ are denoted $\underline{\mathrm{d}\psi^+} = \text{axial}(\mathrm{d}\underline{\underline{R}}\,\underline{\underline{R}}^T)$ and $\underline{\mathrm{d}\psi^*} = \text{axial}(\underline{\underline{R}}^T \mathrm{d}\underline{\underline{R}}) = \underline{\underline{R}}^T \underline{\mathrm{d}\psi^+}$ when resolved in the same bases, respectively. The differential motion vector of point **L** is now defined as

$$\underline{\mathrm{d}\mathcal{U}_\ell^*} = \begin{Bmatrix} \underline{\underline{R}}^T \mathrm{d}\underline{u}^+ \\ \underline{\underline{R}}^T \underline{\mathrm{d}\psi^+} \end{Bmatrix}. \tag{6}$$

Consider now the following frame change operation

$$\underline{\mathrm{d}\mathcal{U}_\ell^+} = \begin{Bmatrix} \mathrm{d}\underline{u}^+ + \tilde{u}^+ \underline{\mathrm{d}\psi^+} \\ \underline{\mathrm{d}\psi^+} \end{Bmatrix} = \underline{\underline{\mathcal{C}}}(\underline{u}^+, \underline{\underline{R}}^+) \underline{\mathrm{d}\mathcal{U}_\ell^*}, \tag{7}$$

where $\underline{\underline{R}}^+ = \underline{\underline{R}}$. This change of frame operation involves two consecutive operations: a change of basis followed by a change of reference point. This dual role of the motion tensor is particularly obvious when it is expressed as a product, $\underline{\underline{\mathcal{C}}} = \underline{\underline{\mathcal{T}}}\,\underline{\underline{\mathcal{R}}}$, where the rotation tensor, $\underline{\underline{\mathcal{R}}}$, performs the change of basis operation, and the translation tensor, $\underline{\underline{\mathcal{T}}}$, the change of reference point operation.

The first three components of the differential motion vector, $\mathrm{d}\underline{u}^+ + \tilde{u}^+ \underline{\mathrm{d}\psi^+}$, represent the components of the differential displacement of the point of handle ℓ that instantaneously coincides with the origin of reference frame $\mathcal{F}_0$ resolved in basis $\mathcal{B}_0$. The next three components are the components of the differential rotation vector resolved in the same basis. Notation $(\cdot)_\ell$ indicates that $\underline{\mathrm{d}\mathcal{U}_\ell^+}$ represents the differential motion of the point of handle ℓ that instantaneously coincides with the origin of reference frame $\mathcal{F}_0$.

3.2 Frame change for the applied load vector

The load externally applied at point **L**, denoted $\underline{\mathcal{A}}_\ell$, was defined in eq. (1). These applied force and moment vectors are now resolved in basis $\mathcal{B}^\ell$ to form $\underline{\mathcal{A}}_\ell^{*T} = \left\{\underline{F}_\ell^{*T}, \underline{M}_\ell^{*T}\right\}$. The following change of frame operation is now considered

$$\underline{\mathcal{A}}_\ell^+ = \begin{Bmatrix} \underline{F}_\ell^+ \\ \underline{M}_\ell^+ + \tilde{u}^+ \underline{F}_\ell^+ \end{Bmatrix} = \underline{\underline{\mathcal{C}}}^{-T}(\underline{u}^+, \underline{\underline{R}}^+) \underline{\mathcal{A}}_\ell^*. \tag{8}$$

This change of frame operation involves two consecutive operations: a change of basis followed by a change of reference point. This dual role of the motion tensor is particularly obvious when it is expressed as a product, $\underline{\underline{\mathcal{C}}}^{-T} = \underline{\underline{\mathcal{T}}}^{-T}\underline{\underline{\mathcal{R}}}$, where the rotation tensor, $\underline{\underline{\mathcal{R}}}$ (equal to $\underline{\underline{\mathcal{R}}}^{-T}$ because $\underline{\underline{\mathcal{R}}}$ is an orthogonal tensor), performs the change of basis operation, and the translation tensor, $\underline{\underline{\mathcal{T}}}^{-T}$, the change of reference point operation.

The first three components of the load vector are the components of the force vector resolved in basis $\mathcal{B}_0$. The next three components, $\underline{M}_\ell^+ + \tilde{u}^+ \underline{F}_\ell^+$, represent the moment applied to handle ℓ computed with respect to the point of the rigid body that instantaneously coincides with the origin of the reference frame, $\mathcal{F}_0$, resolved in basis $\mathcal{B}_0$.

Note the parallel between vector $\underline{\mathcal{A}}_\ell^+$ and the second Piola-Kirchhoff stress tensor [13]. Indeed, $\underline{\mathcal{A}}_\ell^+$ represents the true loads applied to handle ℓ in its deformed configuration, but transferred to the original location of their application point in the reference configuration. Loads $\underline{\mathcal{A}}_\ell^+$ and $\underline{\mathcal{A}}_\ell^*$ form a set of equipollent loads applied to handle ℓ. The change of frame operation described by eq. (8), expresses, in fact, a condition of equipollence.

4 Deformation measures

Consider the configuration of the flexible joint for scenario ℓ illustrated in fig. 2. First, the differential work done by the applied load will be evaluated in section 4.1, and the nature of deformation measures is further discussed in section 4.2. The requirements for the invariance of the deformation measures are established in section 4.3. It is then shown that invariant deformation measures, see sections 4.5 and 4.6, must be based on the tensorial parameterization of motion discussed in section 4.4.

4.1 Flexible joint differential work

The differential work, $\mathrm{d}W$, done by the forces applied to the joint is

$$\mathrm{d}W = \underline{F}_\ell^{+T}\mathrm{d}\underline{u}^+ + \underline{M}_\ell^{+T}\mathrm{d}\underline{\psi}^+ = \underline{\mathcal{A}}_\ell^{*T}\mathrm{d}\underline{\mathcal{U}}_\ell^* = \underline{\mathcal{A}}_\ell^{+T}\mathrm{d}\underline{\mathcal{U}}_\ell^+, \tag{9}$$

where the last two equalities follow from eqs. (7) and (8), respectively. Because handle k does not move, the forces and moments applied at point $\mathbf{K}$ do not work.

Let $\underline{\mathcal{E}}_\ell^+$ be a set of six generalized coordinates that uniquely define the configuration of handle ℓ, *i.e.*, a one-to-one mapping is assumed to exist between these generalized coordinates and the configuration of handle ℓ. It then follows that a one-to-one mapping must exist between the handle's differential motion and increments of the generalized coordinates

$$\mathrm{d}\underline{\mathcal{U}}_\ell^* = \underline{\underline{\mathcal{H}}}^*(\underline{\mathcal{E}}_\ell^+)\mathrm{d}\underline{\mathcal{E}}_\ell^+, \quad \mathrm{d}\underline{\mathcal{U}}_\ell^+ = \underline{\underline{\mathcal{H}}}(\underline{\mathcal{E}}_\ell^+)\mathrm{d}\underline{\mathcal{E}}_\ell^+. \tag{10}$$

Matrix $\underline{\underline{\mathcal{H}}}(\underline{\mathcal{E}}_\ell^+)$ is the Jacobian matrix or tangent operator of the coordinate transformation.

The differential work done by the forces applied to the joint, eq. (9), now becomes

$$\mathrm{d}W = \underline{\mathcal{A}}_\ell^{*T}\underline{\underline{\mathcal{H}}}^*(\underline{\mathcal{E}}_\ell^+)\mathrm{d}\underline{\mathcal{E}}_\ell^+ = \underline{\mathcal{A}}_\ell^{+T}\underline{\underline{\mathcal{H}}}(\underline{\mathcal{E}}_\ell^+)\mathrm{d}\underline{\mathcal{E}}_\ell^+ = \underline{\mathcal{L}}_\ell^{+T}\mathrm{d}\underline{\mathcal{E}}_\ell^+, \tag{11}$$

where the generalized forces associated with the generalized coordinates are defined as

$$\underline{\mathcal{L}}_\ell^+ = \underline{\underline{\mathcal{H}}}^{*T}(\underline{\mathcal{E}}_\ell^+)\underline{\mathcal{A}}_\ell^* = \underline{\underline{\mathcal{H}}}^T(\underline{\mathcal{E}}_\ell^+)\underline{\mathcal{A}}_\ell^+. \tag{12}$$

It is now assumed that the flexible joint is made of an elastic material [14], which implies that the generalized forces can be derived from a potential, the strain energy of the joint, denoted A,

$$\underline{\mathcal{L}}_\ell^+ = \frac{\partial A(\underline{\mathcal{E}}_\ell^+)}{\partial \underline{\mathcal{E}}_\ell^+}. \tag{13}$$

The differential work now becomes

$$\mathrm{d}W = \mathrm{d}\underline{\mathcal{E}}_\ell^{+T}\frac{\partial A(\underline{\mathcal{E}}_\ell^+)}{\partial \underline{\mathcal{E}}_\ell^+} = \mathrm{d}(A), \tag{14}$$

and can be expressed as the differential of a scalar function, the strain energy.

The reasoning presented in this section could be repeated for scenario k. Because scenarios k and ℓ only differ by a rigid body motion, identical results should be obtained. In particular, the differential work for the two scenarios should be identical, leading to $\mathrm{d}W = \underline{\mathcal{A}}_\ell^{+T}\mathrm{d}\underline{\mathcal{U}}_\ell^+ = \underline{\mathcal{A}}_k^{+T}\mathrm{d}\underline{\mathcal{U}}_k^+$. Loading $\underline{\mathcal{A}}_\ell^+$ and $\underline{\mathcal{A}}_k^+$ are referred to the same point, the origin of frame $\mathcal{F}_0$, and expressed in the same basis, $\mathcal{B}_0$; Newton's first law then implies $\underline{\mathcal{A}}_\ell^+ + \underline{\mathcal{A}}_k^+ = 0$, leading to the intuitive result that

$$\mathrm{d}\underline{\mathcal{U}}_k^+ = -\mathrm{d}\underline{\mathcal{U}}_\ell^+. \tag{15}$$

4.2 The deformation measures

In the previous section, quantities $\underline{\mathcal{E}}_\ell^+$ were defined as "a set of generalized coordinates that uniquely define the configuration of handle ℓ," but were otherwise left undefined. For scenario ℓ, the configuration of handle ℓ defines the deformation of the elastic body, and hence, these generalized coordinates are, in fact, deformation measures for the flexible joint. The following notation is introduced

$$\underline{\mathcal{E}}_\ell^+ = \begin{Bmatrix} \underline{\epsilon}^+ \\ \underline{\kappa}^+ \end{Bmatrix}. \tag{16}$$

The first three components of this array form the stretch vector, denoted $\underline{\epsilon}$, and the last three the wryness vector, denoted $\underline{\kappa}$. Both quantities are assumed to form first order tensors.

Because the deformation measures uniquely define the configuration of handle ℓ relative to handle k, the motion tensor, $\underline{\underline{\mathcal{C}}}(\underline{u}, \underline{\underline{R}})$, can be expressed as $\underline{\underline{\mathcal{C}}} = \underline{\underline{\mathcal{C}}}(\underline{\mathcal{E}}_\ell^+)$. It follows that the deformation measures form a parameterization of the motion tensor. In general, the deformation measures are nonlinear functions of six quantities, the three components of the relative displacement vector, $\underline{u}$, and the three parameters that define the relative rotation tensor, $\underline{\underline{R}}$.

For instance, the stretch vector could be selected as the position vector of point $\mathbf{L}$, $\underline{\epsilon}^+ = \underline{u}^+$; note that $\underline{\epsilon}^* = \underline{\underline{R}}^T \underline{\epsilon}^+$, as expected from the tensorial nature of the stretch vector. The Euler angles associated with rotation tensor $\underline{\underline{R}}$ form a valid set of generalized coordinates to characterize the angular motion of handle ℓ, but cannot be the components of the wryness vector because Euler angles do not form the components of a vector. Any vectorial parameterization of rotation [15] is a suitable choice for the wryness vector.

4.3 Change of reference frame

The results derived in section 4.1 were based on the selection of an arbitrary reference frame, $\mathcal{F}_0$, defined by coincident reference points, $\mathbf{K} = \mathbf{L}$, which are material points of handles k and ℓ, respectively, and an orthonormal basis, $\mathcal{B}_0$. Another reference frame could have been selected, $\mathcal{F}_0' = \left[\mathbf{K}' = \mathbf{L}', \mathcal{B}_0' = (\bar{b}_{01}', \bar{b}_{02}', \bar{b}_{03}')\right]$; figure 3 shows the new reference points, $\mathbf{K'} = \mathbf{L'}$, which are material points of handles k and ℓ, respectively. For clarity, the new basis, $\mathcal{B}_0'$, is not shown on the figure. The position vector of point $\mathbf{K'}$ with respect to point $\mathbf{K}$ is denoted $\underline{s}$ and the relative rotation tensor of basis $\mathcal{B}_0'$ with respect to basis $\mathcal{B}_0$ is denoted $\underline{\underline{S}}$. The motion tensor that brings frame to $\mathcal{F}_0$ to $\mathcal{F}_0'$ is denoted $\underline{\underline{\mathcal{C}}}'(\underline{s}, \underline{\underline{S}})$.

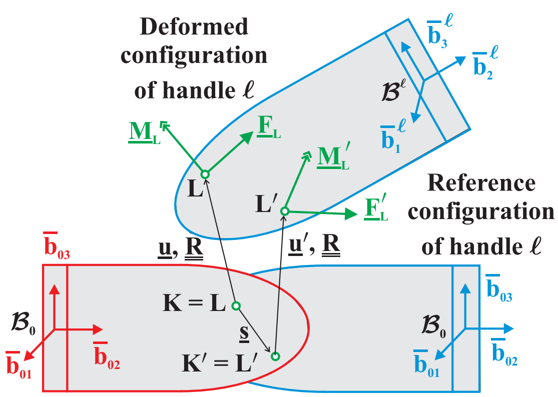

Figure 3: Configuration of the flexible joint for scenario ℓ. For clarity of the figure, the elastic body is not shown

The development presented in section 4.1 could now be repeated for this new choice of basis and reference points, leading to a new set of applied loads, $\underline{\mathcal{A}}_\ell'^+$, deformation measures, $\underline{\mathcal{E}}_\ell'^+$, tangent operator, $\underline{\underline{\mathcal{H}}}(\underline{\mathcal{E}}_\ell'^+)$, and associated generalized forces, $\underline{\mathcal{L}}_\ell'^+$.

If the same problem is treated with scenario ℓ in the two different frames, the sets of loads applied at points **L** and **L'** must be *equipollent*, which implies

$$\underline{\mathcal{A}}_\ell^+ = \underline{\underline{\mathcal{C}}}'^{-T}(\underline{s}, \underline{\underline{S}})\underline{\mathcal{A}}_\ell'^+. \tag{17}$$

This equation expresses the relationship between the components of the loading vector in the two frames, $\mathcal{F}_0$ to $\mathcal{F}_0'$, *i.e.*, the equipollence condition implies that the loading vector is a first-order tensor, see eq. (8).

To be physically meaningful, the corresponding deformation measures must also be invariant with respect to a change of frame, *i.e.*, they must also be first-order tensors, and their components in two frames, $\mathcal{F}_0$ and $\mathcal{F}_0'$, denoted $\underline{\mathcal{E}}_\ell^+$ and $\underline{\mathcal{E}}_\ell'^+$, respectively, must transform according to the rules of transformation for kinematic quantities given by eq. (7), *i.e.*,

$$\underline{\mathcal{E}}_\ell^+ = \underline{\underline{\mathcal{C}}}'(\underline{s}, \underline{\underline{S}})\underline{\mathcal{E}}_\ell'^+, \tag{18}$$

This equation expresses the desired invariance of the deformation measure.

The motion tensor is a second-order tensor and the deformation measure is a parameterization of this motion tensor. According to eq. (18), this parameterization must be a first-order tensor. This implies that the deformation measure must be a tensorial parameterization of motion.

4.4 Parameterization of motion

While the parameterization of rotation has received wide attention [16, 17, 18, 19, 15], much less emphasis has been placed on that of motion [20, 12]. Bauchau *et al.* [21, 9] have proposed the tensorial parameterization of motion, which consists of a motion parameters vector, $\underline{\mathcal{E}}$, that parameterizes the motion tensor, $\underline{\underline{\mathcal{C}}} = \underline{\underline{\mathcal{C}}}(\underline{\mathcal{E}})$. Bauchau and Li [9] have studied the parameterization of motion, with special emphasis on its tensorial nature. They presented a formal proof that motion parameters vectors are first-order tensors if and only if they are parallel to the eigenvectors of the motion tensor associated with its unit eigenvalues.

Let $\underline{\underline{\mathcal{C}}}'$ be a motion tensor; the triple tensor multiplication, $\underline{\underline{\mathcal{C}}}'^{-1}\underline{\underline{\mathcal{C}}}(\underline{\mathcal{E}}_\ell^+)\underline{\underline{\mathcal{C}}}'$, then expresses a change of frame operation for this tensor. The vectorial parameterization of motion [21, 9] presents two fundamental properties. *(1) The vectorial parameterization of motion is tensorial in nature,* as expressed by the following equivalence, see eq. $(55)_{[9]}$[1],

$$\underline{\underline{\mathcal{C}}}(\underline{\mathcal{E}}_\ell'^+) = \underline{\underline{\mathcal{C}}}'^{-1}\underline{\underline{\mathcal{C}}}(\underline{\mathcal{E}}_\ell^+)\underline{\underline{\mathcal{C}}}' \Leftrightarrow \underline{\mathcal{E}}_\ell'^+ = \underline{\underline{\mathcal{C}}}'^{-1}\underline{\mathcal{E}}_\ell^+. \tag{19}$$

The tensorial nature of the second-order motion tensor implies and is implied by the tensorial nature of the motion parameter vector, a first-order tensor. *(2)* Motion parameter vectors are parallel to an *eigenvector of the motion tensor* associated with its unit eigenvalue.

Because these two properties were shown to imply each other, either can be taken as the definition of the vectorial parameterization of motion. A parameterization of motion is tensorial if and only if the motion parameter vector is parallel an eigenvector of the motion tensor associated with its unit eigenvalue. The left-hand side of equivalence (19) expresses the transformation rule for the second-order motion tensor, and the right-hand side of the same equivalence expresses the transformation rule for the first-order motion parameter vector.

An additional property of the vectorial parameterization of motion is the tensorial nature of the associated tangent operator, see eq.$(77)_{[9]}$,

$$\underline{\underline{\mathcal{H}}}(\underline{\mathcal{E}}_\ell'^+) = \underline{\underline{\mathcal{C}}}'^{-1}\underline{\underline{\mathcal{H}}}(\underline{\mathcal{E}}_\ell^+)\underline{\underline{\mathcal{C}}}' \Leftrightarrow \underline{\mathcal{E}}_\ell'^+ = \underline{\underline{\mathcal{C}}}'^{-1}\underline{\mathcal{E}}_\ell^+. \tag{20}$$

[1]The notation eq. $(55)_{[9]}$ refers to eq. (55) of ref. [9]

In summary, the desired invariance of the deformation measure expressed by eq. (18) is achieved if and only if the deformation measure is selected to be the tensorial parameterization of motion, which further implies that the deformation measure is parallel to the eigenvector of the motion tensor associated with its unit eigenvalue.

4.5 Deformation measure invariance

Equation (9) expresses the invariance of the differential work with respect to a change of frame. The equipollence condition of the applied load is expressed by eq. (17) and introducing this condition into eq. (9) yields

$$\mathrm{d}\underline{\mathcal{U}}_\ell^+ = \underline{\underline{\mathcal{C}}}'(\underline{s}, \underline{\underline{S}})\mathrm{d}\underline{\mathcal{U}}_\ell'^+. \tag{21}$$

The equipollence of the applied load and invariance of the differential work imply that the components of the loading vector transform according to the first-order tensor transformation rule expressed by eq. (8) for loading quantities and the components of the differential displacement vector according to that expressed by eq. (7) for kinematic quantities.

Introducing eq. (21) into eq. (10) and pre-multiplying by $\underline{\underline{\mathcal{C}}}'^{-1}$ yields $\mathrm{d}\underline{\mathcal{U}}_\ell'^+ = \underline{\underline{\mathcal{C}}}'^{-1}\underline{\underline{\mathcal{H}}}(\underline{\mathcal{E}}_\ell^+)\underline{\underline{\mathcal{C}}}'\underline{\underline{\mathcal{C}}}'^{-1}\mathrm{d}\underline{\mathcal{E}}_\ell^+$, where $\underline{\underline{\mathcal{C}}}'\underline{\underline{\mathcal{C}}}'^{-1} = \underline{\underline{\mathcal{I}}}$. Introducing eq. (20) then yields $\mathrm{d}\underline{\mathcal{U}}_\ell'^+ = \underline{\underline{\mathcal{H}}}(\underline{\mathcal{E}}_\ell'^+)\underline{\underline{\mathcal{C}}}'^{-1}\mathrm{d}\underline{\mathcal{E}}_\ell^+$, which leads to the expected transformation rule for the components of the differential deformation measure

$$\mathrm{d}\underline{\mathcal{E}}_\ell^+ = \underline{\underline{\mathcal{C}}}'\mathrm{d}\underline{\mathcal{E}}_\ell'^+. \tag{22}$$

The invariance of the differential work written in the form of eq. (11) requires $\mathrm{d}W = \underline{\mathcal{L}}_\ell^{+T}\mathrm{d}\underline{\mathcal{E}}_\ell^+ = \underline{\mathcal{L}}_\ell'^{+T}\mathrm{d}\underline{\mathcal{E}}_\ell'^+$. Introducing eq. (22) then yields

$$\underline{\mathcal{L}}_\ell^+ = \underline{\underline{\mathcal{C}}}'^{-T}\underline{\mathcal{L}}_\ell'^+. \tag{23}$$

In summary, the formulation developed in section 4.1, is frame invariant. Under a change of frame, the components of the applied and generalized loads transform according to eqs. (17) and (23), respectively. The components of the deformation measure, differential displacement, and differential deformation measure transform according to eqs. (18), (21), and (22), respectively. These energetically conjugate first-order tensors present different transformation rules under a change of frame to guarantee the required invariance of the differential work.

The invariance of the various quantities involved in the formulation stems from the tensorial nature of the deformation measure. Because this measure is selected to be the tensorial parameterization of motion, it must be an eigenvector of the motion tensor, *i.e.*, $\underline{\mathcal{E}}_\ell^+ = \underline{\underline{\mathcal{C}}}\,\underline{\mathcal{E}}_\ell^+$. Since the deformation measure is a kinematic quantity, $\underline{\mathcal{E}}_\ell^+ = \underline{\underline{\mathcal{C}}}\,\underline{\mathcal{E}}_\ell^*$, and it follows that $\underline{\mathcal{E}}_\ell^+ = \underline{\mathcal{E}}_\ell^*$, *i.e.*, the components of the deformation measure are identical in frames $\mathcal{F}_0$ and $\mathcal{F}^\ell$. This implies that the deformation measure is identical when viewed by observers in frames $\mathcal{F}_0$ or $\mathcal{F}^\ell$. Consequently, the deformation measure is not biased towards one of the nodes of the joint, a shortcoming of many of the formulation presently implemented in research and commercial codes.

Equation (15) implies $\underline{\mathcal{E}}_k^+ = -\underline{\mathcal{E}}_\ell^+$, which simply corresponds to a sign convention. Henceforth, notation $\underline{\mathcal{E}} = \underline{\mathcal{E}}_\ell^+ = \underline{\mathcal{E}}_\ell^*$ is used, which emphasizes the intrinsic nature of the deformation measure; of course, a change of sign is required for scenario k. Finally, eq. (13) implies $\underline{\mathcal{L}} = \underline{\mathcal{L}}_\ell^+ = \underline{\mathcal{L}}_\ell^*$, which shows the intrinsic nature of the generalized forces; here again, a change of sign is required for scenario k.

4.6 Explicit expression of the deformation measures

The proposed deformation measures are parallel to the eigenvector of the motion tensor associated with its unit eigenvalue. Because this eigenvalue has a multiplicity of two [10], two linearly independent eigenvectors exist, and the deformation measure is a linear combination of these two eigenvectors. An explicit expression of the deformation measure, see eq. (56)[9], is

$$\underline{\mathcal{E}} = \begin{Bmatrix} \underline{\epsilon} \\ \underline{\kappa} \end{Bmatrix} = \begin{Bmatrix} \underline{\underline{D}}(\underline{\kappa})\underline{u} \\ \underline{\kappa} \end{Bmatrix}, \tag{24}$$

where the stretch vector, $\underline{\epsilon}$, is related to the displacement vector, $\underline{u}$, of the handle, the wryness vector, $\underline{\kappa}$, is the vectorial parameterization of rotation [15], and tensor $\underline{\underline{D}}$ is defined by eq. (25).

According to Euler's theorem, the relative rotation of basis $\mathcal{B}^\ell$ with respect to basis $\mathcal{B}_0$ can be expressed as a single rotation of magnitude ϕ about unit vector $\bar{n}$, and the rotation parameter vector is then $\underline{\kappa} = \kappa(\phi)\bar{n}$, where $\kappa(\phi) = \|\underline{\kappa}\|$ is an odd function of angle ϕ.

Tensor $\underline{\underline{D}}$, see eq. (57)[9], is defined as

$$\underline{\underline{D}}(\underline{\kappa}) = \delta_0 \underline{\underline{I}} - \frac{1}{2}\tilde{\kappa} + \delta_2 \tilde{\kappa}\tilde{\kappa}, \tag{25}$$

where functions $\delta_0(\phi)$ and $\delta_2(\phi)$ are given by

$$\delta_0 = \frac{\alpha}{\nu}, \tag{26a}$$

$$\delta_2 = \frac{1}{\kappa^2}\left(\delta_0 - \frac{1}{\varepsilon}\right). \tag{26b}$$

Two even functions of the rotation angle play an important role in the vectorial parameterization of rotation,

$$\nu = \frac{2\sin\phi/2}{\kappa}, \tag{27a}$$

$$\varepsilon = \frac{2\tan\phi/2}{\kappa} = \frac{\nu}{\cos\phi/2}. \tag{27b}$$

Finally, α is an arbitrary parameter that reflects the non-uniqueness of the eigenvector of the motion tensor associated with its unit eigenvalue.

The deformation measure vector is not yet fully defined yet because it depends on the choice of the generating function, $\kappa(\phi)$, of the vectorial parameterization of rotation and furthermore, parameter α can be selected arbitrarily.

Generating functions must be odd functions of the rotation angle; furthermore, for small relative rotations, the wryness vector, $\underline{\kappa}$, should approach the infinitesimal rotation vector, *i.e.*,

$$\lim_{\phi\to 0} \kappa(\phi) = \phi. \tag{28}$$

Similarly, the stretch vector, $\underline{\epsilon}$, should approach the infinitesimal displacement vector when $\underline{u} \to 0$ and $\phi \to 0$. In view of eq. (24), this requirement implies $\lim_{\phi\to 0} \underline{\underline{D}}(\underline{\kappa}) = \underline{\underline{I}}$, or $\lim_{\phi\to 0} \alpha/\nu = 1$, and finally

$$\lim_{\phi\to 0} \alpha(\phi) = 1. \tag{29}$$

In summary, eqs. (24) define the proposed deformation measures for flexible joints. These equations are, in fact, the nonlinear deformation-displacement relationships of the problem. They are not fully determined because two arbitrary scalar functions appear in their definition. The generating function, $\kappa(\phi)$, is an odd function of the rotation angle and must satisfy the limit condition expressed by eq. (28). Parameter α reflects the non-uniqueness of the eigenvector of the motion tensor associated with its unit eigenvalue; it is an even function of the rotation angle and must satisfy the limit condition expressed by eq. (29). Specific choices of these two scalar functions result in families of deformation measures.

5 Flexible joint constitutive laws

The strain energy of the flexible joint is assumed to be a quadratic function of the deformation measures, $A = 1/2\ \underline{\mathcal{E}}^T \underline{\underline{K}}\,\underline{\mathcal{E}}$, where $\underline{\underline{K}}$ is the joint's stiffness matrix for infinitesimal deformations. The generalized forces now become $\underline{\mathcal{L}} = \underline{\underline{K}}\,\underline{\mathcal{E}}$, and eq. (12) then yields

$$\underline{\mathcal{A}}_\ell^+ = \underline{\underline{\mathcal{H}}}^{-T}(\underline{\mathcal{E}})\underline{\underline{K}}\,\underline{\mathcal{E}}, \tag{30a}$$

$$\underline{\mathcal{A}}_\ell^* = \underline{\underline{\mathcal{H}}}^{*-T}(\underline{\mathcal{E}})\underline{\underline{K}}\,\underline{\mathcal{E}}. \tag{30b}$$

Due to the presence of the tangent tensor, the load-deformation relationships are nonlinear, and the deformation-displacement relationships, eqs. (24), are also nonlinear.

The loads applied to handle ℓ resolved in basis $\mathcal{B}_0$, denoted $\underline{\mathcal{A}}$, are $\underline{\mathcal{A}} = \underline{\underline{\mathcal{T}}}^T \underline{\mathcal{A}}_\ell^+ = \underline{\underline{\mathcal{R}}}\,\underline{\mathcal{A}}_\ell^*$. The joint's constitutive laws now become

$$\underline{\mathcal{A}} = \begin{bmatrix} \underline{\underline{F}}^{-T}(\underline{\kappa}) & \underline{\underline{0}} \\ -\underline{\underline{H}}^{-T}(\underline{\kappa})\underline{\underline{L}}^T(\underline{\epsilon},\underline{\kappa})\underline{\underline{F}}^{-T}(\underline{\kappa}) & \underline{\underline{H}}^{-T}(\underline{\kappa}) \end{bmatrix} \underline{\underline{K}}\,\underline{\mathcal{E}}. \tag{31}$$

where tensors $\underline{\underline{F}}$, $\underline{\underline{H}}$, and $\underline{\underline{L}}$ are defined by eqs. (59)[9], (20)[9], and (63)[9], respectively.

Finally, inversion of this equation gives the constitutive laws in compliance form as

$$\underline{\mathcal{E}} = \underline{\underline{S}} \begin{bmatrix} \underline{\underline{F}}^T(\underline{\kappa}) & \underline{\underline{0}} \\ \underline{\underline{L}}^T(\underline{\epsilon},\underline{\kappa}) & \underline{\underline{H}}^T(\underline{\kappa}) \end{bmatrix} \underline{\mathcal{A}}, \tag{32}$$

where $\underline{\underline{S}} = \underline{\underline{K}}^{-1}$ is the compliance matrix for infinitesimal deformations. Given the externally applied loads, $\underline{\mathcal{A}}$, this nonlinear equation yield the joints deformations, in terms of the stretch vector, $\underline{\epsilon}$, and the wryness vector, $\underline{\kappa}$.

6 Numerical examples

The load-deformation and deformation-configuration relationships developed in this paper will be tested on a number of simple examples involving a flexible beam. Figure 4 shows the beam of length L along unit vector $\bar{b}_{01}$, width b along $\bar{b}_{02}$, and height h along $\bar{b}_{03}$. The beam is made of a homogeneous material of Young's modulus E and shear modulus G. The examples presented below use the following data: $L = 0.6$ m, $b = 5$ mm, $h = 15$ mm, $E = 73$ GPa, and $G = E/(2(1+\nu))$, where $\nu = 0.3$.

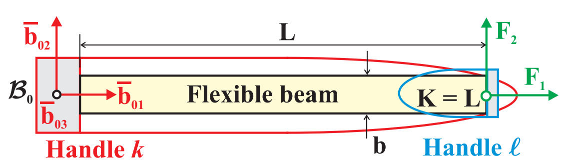

Figure 4: Reference configuration of the flexible beam

Handles k and ℓ are rigidly attached to the root and tip of the beam, respectively. Elementary structural analysis [14] yields the compliance matrix of the joint

$$\underline{\underline{S}} = \begin{bmatrix} \frac{L}{S} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{L^3}{3H_{33}} & 0 & 0 & 0 & \frac{L^2}{2H_{33}} \\ 0 & 0 & \frac{L^3}{3H_{22}} & 0 & -\frac{L^2}{2H_{22}} & 0 \\ 0 & 0 & 0 & \frac{L}{H_{11}} & 0 & 0 \\ 0 & 0 & -\frac{L^2}{2H_{22}} & 0 & \frac{L}{H_{22}} & 0 \\ 0 & \frac{L^2}{2H_{33}} & 0 & 0 & 0 & \frac{L}{H_{33}} \end{bmatrix}, \tag{33}$$

where $S = Ebh$, $H_{22} = Ebh^3/12$, $H_{33} = Ehb^3/12$, and $H_{11} = Ghb^3/3$ are the beam's axial stiffness, bending stiffness with respect the unit vector $\bar{\imath}_2$, bending stiffness with respect the unit vector $\bar{\imath}_3$, and torsional stiffness, respectively.

Various combinations of forces and moments are applied to handle k, and the resulting displacements and rotations are then evaluated using the joint's constitutive laws, eqs. (32). These predictions are compared with those of a finite element solution for geometrically exact beam models [10], which provide an exact treatment of the kinematics of the system, but assume the strains to remain small at all time. This latter assumption is equivalent to assuming a constant compliance matrix, as done here. All the numerical solutions shown below are obtained by modeling the beam with 12 cubic elements, corresponding to a total 216 degrees of freedom.

In the first example, the joint is subjected to a single bending moment about unit vector $\bar{\imath}_3$, denoted M_3. For this simple case, eqs. (32) can be solved analytically to yield $\kappa_3(\phi) = \sigma_0(\phi)\bar{M}_3$, where ϕ is the rotation angle of handle ℓ about unit vector $\bar{\imath}_3$ and $\bar{M}_3 = LM_3/H_{33}$. The displacement components of handle ℓ along unit vectors $\bar{\imath}_1$ and $\bar{\imath}_2$ are then $\bar{u}_1 = u_1/L = -(1 - \cos\phi)/2$ and $\bar{u}_2 = u_2/L = 1/2\ \sin\phi$.

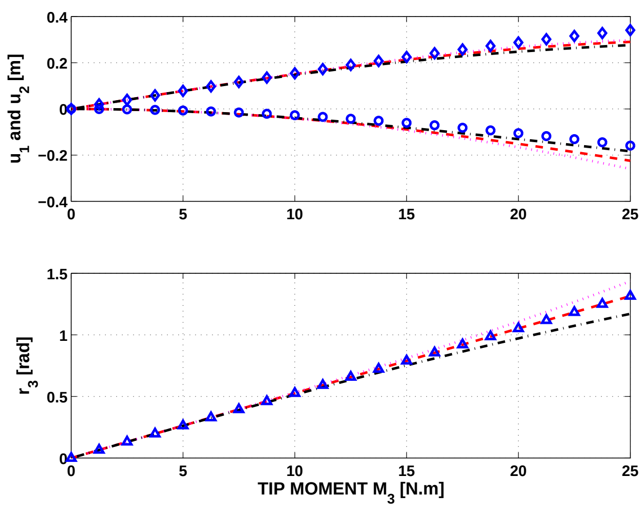

Figure 5: Joint deformation under a single moment. Top figure: displacement components u_1 ($\circ$) and u_2 ($\diamond$); bottom figure: rotation r_3 ($\triangle$). Exact solution: symbols. Present solution: $\kappa(\phi) = \phi$, dashed line; $\kappa(\phi) = 4\sin\phi/4$, dotted line; $\kappa(\phi) = 4\tan\phi/4$, dashed-dotted line

The exact solution of this problem is easily found because the beam deforms into an arc of circle under the single applied moment, leading to $\phi = \bar{M}_3$, $\bar{u}_1 = -(1-\sin\phi/\phi)/2$, and $\bar{u}_2 = (1-\cos\phi)/\phi$, see fig. 5. Three approximate solutions obtained from the proposed approach for three different generating functions, $\kappa(\phi) = \phi$, $\kappa(\phi) = 4\sin\phi/4$, and $\kappa(\phi) = 4\tan\phi/4$, are also depicted in this figure. For $\kappa(\phi) = \phi$, corresponding to the exponential map of rotation, the proposed approach gives the exact solution of the joint's relative rotation. The transverse displacement of the joint is well captured up to very large displacement magnitudes, $u_2 \approx 0.3$ m, for a beam of length $L = 0.6$ m. The beam's foreshortening, a higher-order nonlinear effect, is also well predicted up to large transverse displacements.

If the joint were made of a nonlinear material, the curvature-relative rotation relationship would become nonlinear, and the generating function could be selected to approximate this numerically or experimentally observed behavior as closely as possible. This will enable the present approach to deal with nonlinear elastic manner in an approximate manner. This effect is apparent in fig 5 that depicts the curvature-relative rotation relationship for generating functions $\kappa(\phi) = 4\sin\phi/4$ and $\kappa(\phi) = 4\tan\phi/4$, which give rise to softening or stiffening material behaviors, respectively.

The second example involves the same flexible joint now subjected to two moment components,

$M_2 = 3\lambda$ N·m and $M_3 = \lambda$ N·m, acting about unit vectors $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively, where $\lambda \in [0, 12]$ is the loading factor.

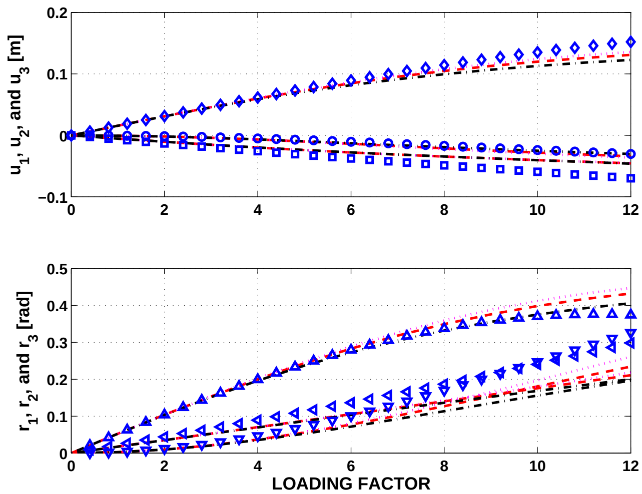

Figure 6: Joint deformation under two moments. Top figure: displacement components u_1 ($\circ$), u_2 ($\diamond$), and u_3 ($\Box$); bottom figure: exponential map components r_1 ($\bigtriangledown$), r_2 ($\triangleleft$), and r_3 ($\bigtriangleup$). Finite element solution: symbols. Present solution: $\kappa(\phi) = \phi$, dashed line; $\kappa(\phi) = 2\sin\phi/2$, dotted line; $\kappa(\phi) = 2\tan\phi/2$, dashed-dotted line

Figure 6 illustrate the ability of the proposed approach to capture the coupled, three-dimensional response of the joint up to large relative displacements and rotations.

In the next example, the joint is subjected to two forces: a constant force $F_2 = 20$ N and a linearly increasing tensile force $F_1 \in [0, 300]$ N, acting along unit vectors $\bar{\imath}_2$ and $\bar{\imath}_1$, respectively. Under the effect of the tensile force, the joint stiffens and the displacement component u_2 resulting from the constant force component F_2 decreases, as shown in fig. 7. Here again, the predictions of the proposed approach are found to be in qualitative agreement with the finite element solution.

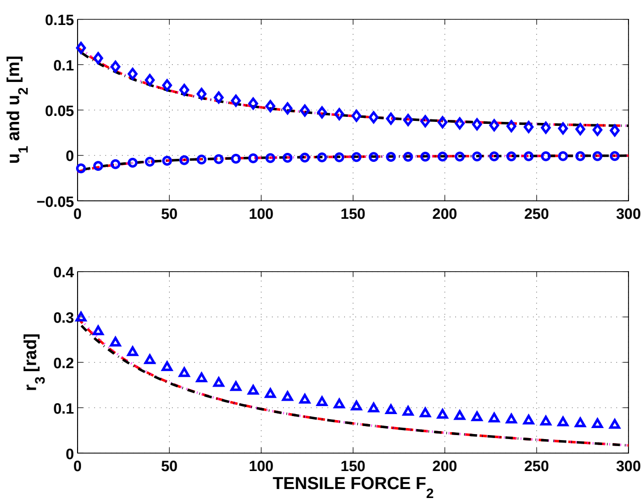

Figure 7: Joint deformation under two forces. Top figure: displacement components u_1 ($\circ$) and u_2 ($\diamond$); bottom figure: exponential map component r_3 ($\bigtriangleup$). Finite element solution: symbols. Present solution: $\kappa(\phi) = \phi$, dashed line; $\kappa(\phi) = 2\sin\phi/2$, dotted line; $\kappa(\phi) = 2\tan\phi/2$, dashed-dotted line

The stiffening of the joint under a tensile force is a nonlinear effect that is captured by the proposed approach because the equilibrium equations of the joint are expressed in the deformed configuration of the system. This prompts the following question: is the proposed formulation able to predict the instability of the joint under compressive load?

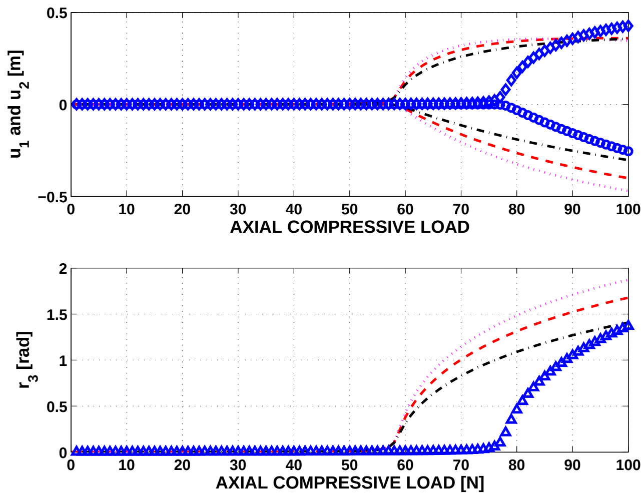

Figure 8: Joint deformation under compressive force. Top figure: displacement components u_1 ($\circ$) and u_2 ($\diamond$); bottom figure: exponential map component r_3 ($\triangle$). Finite element solution: symbols. Present solution: $\kappa(\phi) = \phi$, dashed line; $\kappa(\phi) = 4\sin\phi/4$, dotted line; $\kappa(\phi) = 4\tan\phi/4$, dashed-dotted line

Figure 8 shows the response of the system subjected to a small, constant load $F_2 = 0.1$ N and a linearly increasing compressive load, $F_1 \in [0, 100]$ N. The Euler buckling load of the beam [14] is $P_{\text{Euler}} = \pi^2 H_{33}/(4L^2) = 78$ N, which is accurately predicted by the finite element model. The present model also predicts the buckling phenomenon, although for a lower compressive load of about 60 N. The inaccurate prediction of the present model is due to the fact that it uses 6 degrees of freedom only, in contrast with the 216 degrees of freedom used in the reference solution. Modeling the problem with a single two-node beam element also results in an inaccurate prediction of the buckling load, which is over-predicted by about 50%.

It is also possible to trace the post-buckling path of the system. If a constant load $F_2 = 10$ N and a compressive load $F_1 \in [0, 100]$ N are applied to the joint, it quickly enters the post-buckling regime, as depicted in fig. 9. The proposed model traces the post-buckling path for up to very large displacements and rotations: for a compressive load of 100 N, the relative rotation of the joint is of about 180 degrees.

All the predictions presented in this section are in good qualitative agreement with exact solutions for geometrically exact beams obtained from nonlinear finite element simulations, up to very large relative displacements and rotations of the flexible joint. For small to moderate displacements and rotations, the agreement between the predictions of the proposed formulation and exact solutions is accurate.

It must be emphasized that the present formulation only "knows" the linearized compliance matrix of the joint. The nonlinear governing equations of geometrically exact beams are not derived. Yet, the proposed deformation measures used in conjunction with the linearized compliance matrix provide constitutive laws for the flexible joint that qualitatively describe its behavior up to large relative displacements and rotations. Instabilities, such as buckling under large compressive load or lateral buckling under large transverse loads (not shown here for brevity sake) are also predicted by the proposed formulation. For small displacements and rotations, accurate predictions are obtained.

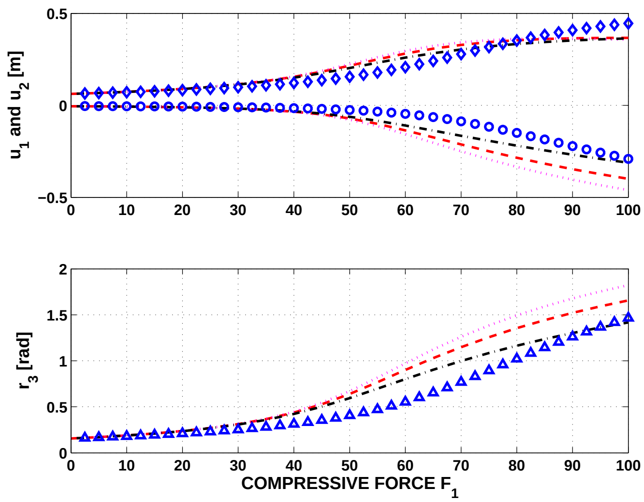

Figure 9: Joint deformation under two forces. Top figure: displacement components u_1 ($\circ$) and u_2 ($\diamond$); bottom figure: exponential map component r_3 ($\triangle$). Finite element solution: symbols. Present solution: $\kappa(\phi) = \phi$, dashed line; $\kappa(\phi) = 2\sin\phi/2$, dotted line; $\kappa(\phi) = 2\tan\phi/2$, dashed-dotted line

7 Conclusions

This paper focused on the definition of appropriate deformation measures for elastic bodies of finite dimension, in contrast with classical strain measures that are defined for infinitesimal elements of an elastic body. It was first argued that to be physically meaningful, these deformation measures must be of a tensorial nature. Next, it was proved that this requirement is satisfied if and only if the deformation measures are parallel to the eigenvector of the motion tensor associated with its unit eigenvalue.

Equipped with these deformation measures, constitutive laws for the flexible joint were derived by assuming the existence of a strain energy function that is a quadratic form of these deformation measures. Because all the quantities involved in the formulation are of a tensorial nature, the behavior of the joint presents the required invariance with respect to changes of basis or reference point. Furthermore, the proposed strain measures are unbiased. Flexible joint formulations described in the literature up to date do not appear to present these desirable characteristics.

While the proposed deformation measures remain tensorial for deformations of arbitrary magnitude, nonlinear constitutive laws should be used if the joint undergoes large deformations. The numerical examples presented in this paper use linear constitutive laws to model a joint consisting of a simple flexible beam. The behavior joint is accurately predicted for small and moderate deformations and the correct qualitative behavior for up to very large displacements and rotations is observed.

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