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FINITE ELEMENT BASED MODAL ANALYSIS OF HELICOPTER ROTOR BLADES

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Abstract - The finite element approach to helicopter blade modeling is increasingly popular, but the cost associated with this analysis technique can be prohibitive when realistic rotor problem are considered. A modal analysis appears attractive, but to be successful, it must be based on expressions containing simple algebraic nonlinearities only, of the lowest possible order. In this paper it is shown that Reissner's Principle, combined with Euler Parameters to represent finite rotations allows an efficient and accurate model of the problem, and the governing equations are cubic only. Furthermore, it is well known that the accuracy of modal methods strongly depends on the "quality" of the assumed modes, i.e., their ability to capture the actual behavior of the blade. Natural vibration modes are often used in modal analyses, however they contain no information about the nonlinear behavior of the blade since they are obtained from the linearized equations of the problem. The use of perturbation modes is proposed in this paper, and is shown to give excellent results for both displacements and internal forces, even for strongly nonlinear problems. Finally, the use of perturbation modes in nonlinear dynamic problem is demonstrated.

1. INTRODUCTION

The versatility and efficiency of the finite element method makes it an attractive tool for the analysis of helicopter rotor blades. Hodges [1] has recently reviewed various finite element approaches, giving a comprehensive discussion of their assumptions and features. An analysis including moderate rotations was developed by Friedmann and Straub [2], as well as by Sivaneri and Chopra [3] based on the formulation of Hodges and Dowell [4]. Giavotto *et al.* [5] and Borri *et al.* [6] developed an approach that includes finite rotations, as well as cross sectional warping deformations. Finally, Bauchau *et al.* [7-9] proposed a model for arbitrarily large displacements and rotations of naturally curved and twisted blades.

These various approaches are very attractive because they allow accurate modeling of rotor blades. The complex kinematics resulting from the large displacements and rotations can be handled in a rational manner, and the intricate elastic behavior of composite blades can be treated realistically by introducing transverse shearing and warping deformations, as well as elastic couplings. However, the cost of such analyses can be prohibitive when realistic problem must be treated.

Consider a composite blade with varying properties along the span: 100 to 150 degrees of freedom (DOFs) are typically necessary to accurately represent its geometry and physical properties. This number might appear small, as problems involving 1,000, or even 10,000 DOFs are routinely solved with large finite element codes. However, in the case of a helicopter blade, the analyst is interested in determining its linear and nonlinear static behavior, its dynamic characteristics i.e. its natural frequencies and mode shapes, its nonlinear, periodic dynamic response, and the stability characteristics of this periodic response. The first two analysis types are relatively straightforward to handle, but the latter ones are far more complex.

Consider the prediction of the nonlinear periodic response of the blade using the finite element element in time method [10]: the total number of DOFs equals the number of DOFs used for the spatial model times the number of time stations. If 64 time stations are used, this will yield 6,400 to 9,600 DOFs to be solved for in an iterative manner, since the problem is nonlinear. For a gimballed rotor, all the blades must be considered simultaneously since they will interact, hence a three bladed gimballed rotor would require the nonlinear solution of 19,200 to 28,800 DOFs, rendering the analysis prohibitively expensive.

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Additional problems will appear when stability analysis is performed using Floquet's theory, which is the standard tool for dealing with the stability of periodic systems. In this approach, the stability of the system is assessed from the eigenvalues of the transition matrix, which is a fully populated matrix of an order equal to twice the number of spatial DOFs. Considering once again the above example, the transition matrix would be of order 200 to 300 for a single blade, and 600 to 900 for the gimballed rotor. Furthermore, this matrix is often ill-conditioned because the characteristic frequencies of the system vary over an extremely wide range. For instance, if six DOFs are considered at each node of the blade, axial frequencies will be included in the model. Such frequencies are many orders of magnitude larger than the first flap frequency of the blade, yielding an ill-conditioned transition matrix. This limitation is inherent to the approach, and will not disappear with increased computational power.

In view of these numerical difficulties, a modal representation appears as a natural way of reducing the number of degrees of freedom of the problem. In fact modal approaches have been very widely used to analyze rotor blades [11], and have the added advantage of involving degrees of freedom that have a direct physical meaning. However, modal approaches are based on an inherent assumption: the motion of the blade is restricted to the superposition of a small number of prescribed modes. Furthermore, when applied to nonlinear problems, there is no assurance of convergence or accuracy of the procedure. The goal of this paper is to develop a finite element based modal analysis for rotor blades. The expression *finite element based* refers to the fact that a conventional finite element model of the blade is subjacent to the modal analysis which accuracy can be assessed by reference to this complete finite element model. In the development of a nonlinear finite element based modal approach, three points are of particular importance: the type of nonlinearities, their order, and the choice of the modes.

Consider for instance a nonlinearity of trigonometric type say $\cos\gamma$, appearing in the strain energy expression (γ is an unknown rotation angle). In the modal approximation, this angle is expanded as $\gamma = \gamma^i\Psi^i$, where γ^i are the assumed mode shapes, Ψ^i the generalized coordinates, and summation over all assumed modes is implied by the repeated indices. To evaluate the strain energy, the expression $\cos\gamma^i\Psi^i$ must then be integrated along the span of the blade; this is of course impossible since Ψ^i are as yet unknown, and due to the transcendental nature of the trigonometric functions. To avoid this problem, it is customary to expand the cosine function in a truncated series: $\cos\gamma^i\Psi^i \simeq 1 - 1/2\ \gamma^i\gamma^j\Psi^i\Psi^j$. This means of course that the analysis will be limited to moderate rotations. Hence, if we wish to develop a modal approximation without introducing additional assumptions, the nonlinearities must be of simple, algebraic type.

Consider next the order of the nonlinearities, say γ^n, where n is the order of the nonlinearity. In the modal expansion this becomes $\gamma^i\gamma^j..\gamma^l\Psi^i\Psi^j..\Psi^l$. It is clear that the number of coefficients generated by such expression is proportional to N^n, where N is the number of assumed modes. Hence for a 12 mode approximation of an expression containing sixth order nonlinearities, $2.9\mathrm{x}10^6$ coefficients will be generated, requiring 22 megabytes of storage on a computer! From this discussion, it is clear that a modal approach must be based on expression containing simple, algebraic type of nonlinearities only, and the order of the nonlinearities must be as low as possible.

Finally, we turn to the choice of modes: in general, natural vibration modes have been selected in modal analyses of rotor blades. The relative merits of various sets of modes have been investigated, for instance, coupled or uncoupled free vibration modes of a rotating or non-rotating blade [12-14]. It is important to note that natural vibration modes characterize the linearized dynamic behavior of the blade, i.e. the dynamic behavior of small, time dependent perturbations about a given, steady equilibrium position of the blade. Even though it is natural to use such modes in the analysis of nonlinear problem, it is well known that the accuracy and efficiency of a modal method depends on the "quality" of the assumed modes, i.e. the ability of the assumed modes to represent the actual response of the blade.

When such modes are used in conjuction with a displacement based energy formulation that includes axial displacements as an independent variable, the performance of the modal approximation is extremely poor. Consider the lateral deflection of a blade in the nonlinear range, under a simple tip oscillatory load. If flapping modes are used in the modal approximation, the lateral deflection is found to be much smaller than that predicted by the full finite element model. This can be explained by the fact that flapping modes contain no axial component (since they are linearized modes), hence foreshortening of the blade is not allowed in the modal approximation and this results in large axial loads which in turn, stiffen the blade considerably. The situation is somewhat improved by adding axial vibration modes, but a large number of these modes is required to obtain a good solution.

The reason for this behavior is twofold. First, in a displacement based formulation, the internal forces are related to the displacement field through the stress-strain relationships. Hence, a very small error in the estimation of the axial strain (as should be expected from a modal approximation) will result in very large axial forces, because of the very large axial stiffness of the blade. In fact, the inextensibility assumption is often made to avoid this problem, however, the formulation is then restricted to single load path blades. In this paper, we will overcome this problem by using a formulation based on Reissner's Principle [15,16], for which displacements and internal forces are independent variables.

Second, the actual axial displacement of the blade is due to foreshortening (a nonlinear kinematic phenomena), whereas axial vibration modes characterize true axial vibrations (a purely linear vibratory phenomenon). In other words, we are trying to "synthesize" a nonlinear kinematic mode shape, with linear vibratory mode. These two phenomena are not physically related, hence, we should hardly expect obtaining good results in predicting axial displacements. This clearly shows the need for selecting alternate mode shapes that contain information about the nonlinear behavior of the structure. The concept of perturbation modes was introduced by Thompson and Walker [17], and later used for static analysis by Noor *et al.* [18,19]. In this paper, perturbation modes will be used in both static and dynamic analyses.

The first part of this paper will discuss a finite element formulation for arbitrarily large displacements and rotations of naturally curved and twisted composite rotor blades that includes transverse shearing and torsion related warping deformations. Reissner's Principle will be the basis of the formulation, and Euler Parameters will be used to represent large rotations. It will be shown that the resulting equations only involve cubic nonlinearities, hence, they are ideally suited for a modal approximation that will be introduced in the second part of the paper. Finally perturbation modes will be discussed and numerical results will be presented to demonstrate the efficiency and accuracy of the procedure.

2. GEOMETRY OF THE BLADE AND KINEMATICS OF THE DEFORMATION

Consider the naturally curved and twisted blade depicted in Fig. 1. The triad i_1, i_2, i_3 is fixed in space and the triad e_1, e_2, e_3, is attached to a reference line along the axis of the blade. e_1 is chosen tangent to the reference line, and e_2, e_3 defines the plane of the cross-section. The curvilinear coordinates along this triad are x_1, x_2, and x_3 respectively. r and R are the position vectors of a particle of the blade in the undeformed and deformed positions, respectively, and u is the displacement vector of the reference line. g_1 and G_1 are the base vectors in the undeformed and deformed positions, respectively, and e_1 and E_1 the corresponding base vectors at the reference line. Finally, the triad e_i^* is defined on the deformed cross-section as $e_2^* = E_2$, $e_3^* = E_3$, and $e_1^* = e_2^* \times e_3^*$, based on the assumption of the cross-sections being infinitely rigid in their own plane. The definitions of these various vectors, and their interrelationship are given in [9]. Two rotation matrices t and T are defined as:

$$e_i = t^T(x_1)\, i_i, \text{ and} \tag{1}$$

$$e_i^* = T^T(x_1)\, i_i \; . \tag{2}$$

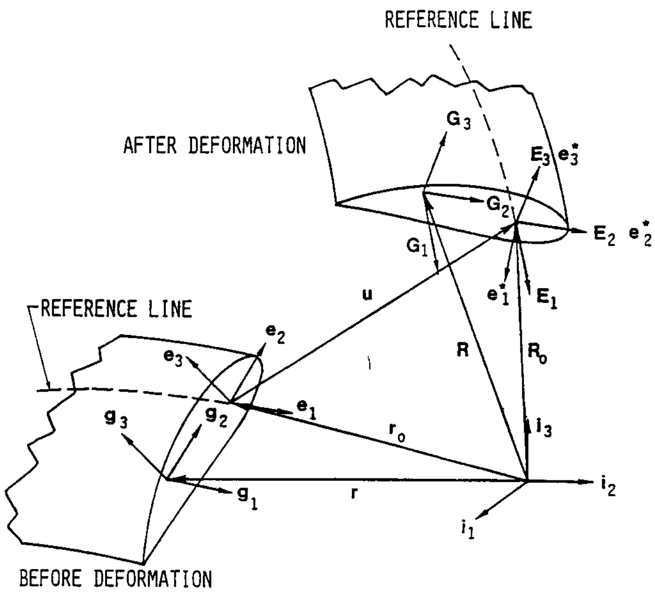

Fig. 1. Geometry of the beam before and after deformation.

3. REISSNER'S PRINCIPLE

In this paper, we focus on thin walled beams which are a realistic model for helicopter blades. For these sections the only nonvanishing stress components are the axial and shear stress flows, n and q, respectively. The corresponding strain components are the axial and shear strains, ε and γ, respectively. For such problem, Reissner's Principle states:

$$\delta \int_0^L \int_\Gamma [(\varepsilon n + \gamma q) - \frac{1}{2}(a_{nn}n^2 + a_{qq}q^2 + 2a_{nq}nq)]\, ds dx_1 - \delta W = 0, \tag{3}$$

where s is the curvilinear coordinate describing the thin-walled contour Γ, L the length of the blade, W the work done by the applied loads, and the compliance coefficients relate strain to stress as:

$$\begin{bmatrix} \varepsilon \\ \gamma \end{bmatrix} = \begin{bmatrix} a_{nn} & a_{nq} \\ a_{nq} & a_{qq} \end{bmatrix} \begin{bmatrix} n \\ q \end{bmatrix} \tag{4}$$

It is important to note that, with this principle, independent assumptions can be made on the strain and stress fields. The strain distribution is assumed as:

$$\varepsilon = e_1 - x_2\kappa_3 + x_3\kappa_2 + \phi_1\kappa_4 - k_1 r\, \phi_1^+ e_4 \tag{5}$$

$$\gamma = x_2^+ e_2 + x_3^+ e_3 + r\, \kappa_1 + \phi_1^+ e_4. \tag{6}$$

where e_1, e_2, e_3, and κ_1, κ_2, κ_3 are the three strains and three curvatures of the blade, respectively; k_1 its pretwist; $r = x_2 x_3^+ - x_3 x_2^+$; ϕ_1 the Saint-Venant torsion related warping function, e_4 the amplitude of this warping deformation and κ_4 its derivative; $(.)^+$ denotes derivative with respect to s. Relation (5) and (6) were derived in [9], and the strain were shown to be related to the displacement as follows:

$$\begin{bmatrix} 1+e_1 \\ e_2 \\ e_3 \end{bmatrix} = T^T \begin{bmatrix} u_1' + t_{11} \\ u_2' + t_{21} \\ u_3' + t_{31} \end{bmatrix}, \text{ and} \tag{7}$$

$$\kappa_i = K_i - k_i, \ (i = 1,2,3), \tag{8}$$

where t_{ij} are the components of the t matrix; $(.)'$ denotes derivative with respect to x_1; K_i and k_i the curvature of the reference line in the deformed and undeformed configurations, respectively (See Appendix I). We now turn to the stress distribution assumed as:

$$\begin{bmatrix} n \\ q \end{bmatrix} = \begin{bmatrix} A_{nn} & A_{nq} \\ A_{nq} & A_{qq} \end{bmatrix} B\, X \tag{9}$$

where $X^T=(X_1,X_2,X_3,X_4,Y_1,Y_2,Y_3,Y_4)$ is a vector of unknown stress parameters; A_{nn}, A_{qq}, and A_{nq} the stiffness coefficient (the inverse of the compliance coefficients of relation (4)); and

$$B = \begin{bmatrix} 1 & 0 & 0 & -k_1 r\phi_1^+ & 0 & x_3 & -x_2 & \phi_1 \\ 0 & \phi_2^+ & \phi_3^+ & \phi_1^+ & r & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

where ϕ_2 and ϕ_3 are the transverse shearing related Saint-Venant warping functions. However, it is preferable to work with stress parameters having a direct physical meaning, such as:

$$F_1 = \int_\Gamma n\, ds;\ F_2 = \int_\Gamma x_2^+ q\, ds,\ F_3 = \int_\Gamma x_3^+ q\, ds,\ F_4 = \int_\Gamma \phi_1^+(q - k_1 r n)\, ds \tag{11}$$

$$M_1 = \int_\Gamma rq\, ds;\ M_2 = \int_\Gamma x_3 n\, ds,\ M_3 = -\int_\Gamma x_2 n\, ds,\ M_4 = \int_\Gamma \phi_1 n\, ds. \tag{12}$$

F_1 is the axial force and M_1 the torque, F_2 and F_3 the shear forces, M_2 and M_3 the bending moments, and finally, F_4 and M_4 are the force and moment, respectively, associated with the torsional warping induced stresses. Introducing (9) into (11) and (12), we find after integration:

$$F = A\, X \tag{13}$$

where $F^T = (F_1,F_2,F_3,F_4,M_1,M_2,M_3,M_4)$, and A is a matrix of cross-sectional coefficients that can be readily obtained. The strain and stress assumptions (5), (6), and (9) are now introduced into (4), and after integration over the cross-section we find:

$$\delta \int_0^L (S^T F - \frac{1}{2} F^T H F)\, dx_1 - \delta W = 0. \quad (14)$$

where $S^T = (e_1, e_2, e_3, e_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$, and H the compliance matrix is given by:

$$H = A^{-T} \{ \int_\Gamma B^T \begin{bmatrix} A_{nn} & A_{nq} \\ A_{nq} & A_{qq} \end{bmatrix} B\, ds \} A^{-1} \quad (15)$$

To complete the formulation, a specific representation of the finite rotation involved in the matrix T must be selected. In this paper, the Euler Parameters [20] are selected as they afford a simple algebraic representation of the rotations (See Appendix 1 for a definition of these parameters). Géradin et al. [21] have shown this approach to be well suited for finite element formulations, however there is a penalty to be payed: the four Euler parameters are not independent as the normality condition (A2) must be satisfied. It is well known that three parameters only are sufficient to represent finite rotations [20], such as Euler type angles, and Rodrigue's or Milenkovic's parameters. However, these representations introduce transcendental functions or denominators, hence they are not suitable for modal approaches. The normality condition could be enforced using a penalty technique, however, in parallel to the Reissner formulation it is preferable to define the following fictitious strain:

$$e_5 = q_0^2 + q_1^2 + q_2^2 + q_3^2 - 1 \quad (16)$$

and a Lagrange mutiplier F_5, then add the term $F_5 e_5 - \frac{1}{2} \alpha F_5^2$ to (14). It is clear that F_5 can be interpreted as the fictitious stress associated with e_5, and by choosing α large enough, e_5 can be made as small as desired. Hence, (14) still holds if F is augmented with F_5, S with e_5, and H with a ninth diagonal term α. In summary, this formulation involves 17 unknown functions: eight displacements (i.e. three translations, four Euler Parameters, and one torsional warping amplitude), and nine internal forces.

A finite element approximation to (14) is relatively straightforward: all displacements are interpolated with cubic polynomials, and all internal forces are interpolated with parabolic polynomials. Internal force continuity is not enforced at interelement boundary, hence, they can be eliminated at the element level. This way of proceding was shown to be most efficient by Noor *et al.* [22].

The soundness of this formulation will be demonstrated with a simple example. The finite element results will be compared with experimental measurements obtained by Dowell and Traybar [23] for a slender, cantilevered beam with a solid aluminum cross-section subjected to a tip load P. The angle Φ_o between the loading direction and the major axis of the cross-section (the "loading angle") can be varied from 0 to 90 degrees in the experimental set-up, yielding a wide range for nonlinear problems where torsion and bending are coupled. Figures 2a, b, and c show the tip twist angle, the flatwise tip displacement (i.e. the displacement perpendicular to the major axis of the cross-section), and the edgewise tip displacement (parallel to the major axis), respectively, versus the loading angle, for three values of the tip load (P = 1,2 and 3 lbs). In all cases excellent correlation is found between the finite element predictions and the experimental results.

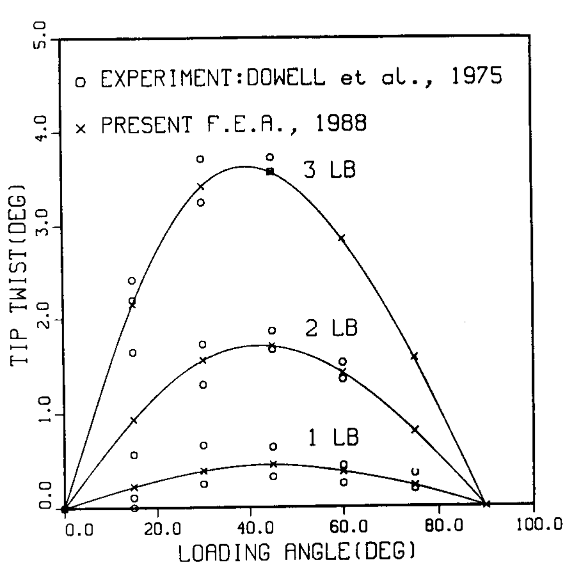

Fig. 2(a). Tip twist angle.

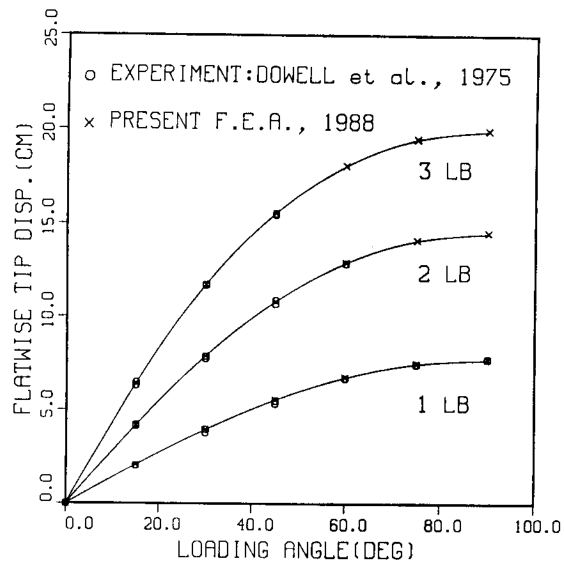

Fig. 2(b). Flatwise displacement.

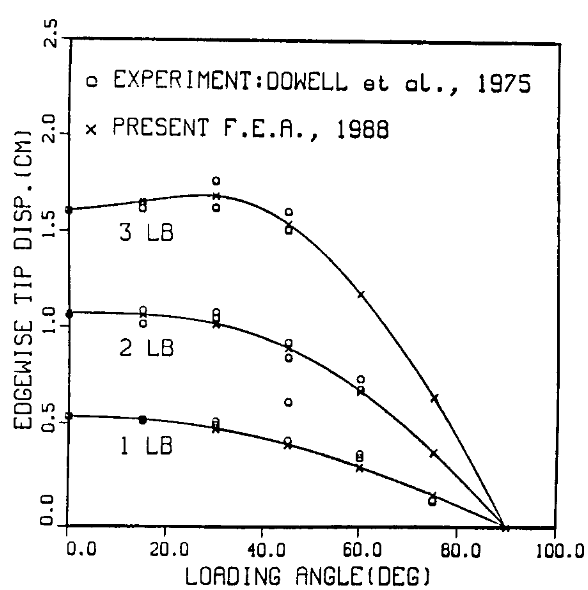

Fig. 2(c). Edgewise displacement.

4) MODAL APPROXIMATION

We now turn our attention to a modal approximation of the problem. The Reissner formulation described in the previous section is well suited for this procedure as it contains simple algebraic nonlinearities only, of the fourth order at most. Indeed, the highest order term are $e_1F_1 + e_2F_2 + e_3F_3$, where the strains are cubic (7) and the forces linear. In the modal approach, we write:

$$u_i = \bar{u}_i + u^j_i \psi^j \quad i = 1,3; \qquad q_i = \bar{q}_i + q^j_i \psi^j \quad i = 1,4; \qquad j = 1,N \qquad F_i = \bar{F}_i + F^j_i \psi^j \quad i = 1,9; \tag{17}$$

where the barred quantities represent a known reference state of the blade about which the modal expansion is performed; u^j_i, q^j_i, and F^j_i the assumed mode shapes for the displacements, Euler parameters, and internal forces, respectively; ψ^j the generalized coordinates; and N the number of assumed mode shapes. Introducing (17) into (14) and integrating over the span of the blade yields:

$$\delta \{ A_4^{ijkl} \psi^i \psi^j \psi^k \psi^l + A_3^{ijk} \psi^i \psi^j \psi^k + A_2^{ij} \psi^i \psi^j + A_1^i \psi^i - W \} = 0, \tag{18}$$

where A_4^{ijkl}, A_3^{ijk}, A_2^{ij}, and A_1^i are coefficients evaluated in the integration process. The resulting algebraic equations clearly are of third order, and are readily solved using a Newton itera- procedure.

5) PERTURBATION MODES

The perturbation modes are derived from the full finite element formulation described in section 3. The incremental form of the equations is:

$$K(\bar{u})\, \Delta u = Q - F(\bar{u}) \tag{19}$$

where u is the vector of nodal unknown that includes both nodal displacements and forces, K the stifness matrix linearized about a reference configuration $\bar{u}$, Q the vector of externally applied loads, and F the vector of equivalent nodal forces. Equilibrium is achieved when $\Delta u = 0$, or

$$F(u^*) - Q = 0, \tag{20}$$

which simply states that at equilibrium, the equivalent nodal forces are equal to the applied loads. The equilibrium configuration u^* is of course a function of the applied load. Consider now an applied load of the form λQ (λ) is a scalar, the equilibrium condition is:

$$F_i\,(u(\lambda)) - \lambda Q_i = 0. \tag{21}$$

Since F_i is a nonlinear function of the displacements, Eq. (21) cannot be solved easily, however, taking the first derivative of this relation with respect to λ yields:

$$\frac{\partial F_i}{\partial u_j} u_j^{(1)} = Q_i \tag{22}$$

where $\frac{\partial F_i}{\partial u_j} = K_{ij}$ is the linearized stiffness matrix, and $u_j^{<i>}$ is the first perturbation mode which is clearly nothing else but the solution of the linearized problem. The second and higher order derivatives become:

$$K_{ij}u_j^{(2)} = -\frac{\partial F_i^2}{\partial u_j \partial u_k} u_j^{(1)} u_k^{(1)} \tag{23}$$

$$K_{ij}u_j^{(3)} = -3\frac{\partial F_i^2}{\partial u_j \partial u_k} u_j^{(2)} u_k^{(1)} - \frac{\partial F_i^3}{\partial u_j \partial u_k \partial u_l} u_j^{(1)} u_k^{(1)} u_l^{(1)} \tag{24}$$

$$K_{ij}u_j^{(4)} = -\frac{\partial F_i^2}{\partial u_j \partial u_k}(4u_j^{(3)}u_k^{(1)} + 3u_j^{(2)}u_k^{(2)}) - 6\frac{\partial F_i^3}{\partial u_j \partial u_k \partial u_l} u_j^{(2)} u_k^{(1)} u_l^{(1)}. \tag{25}$$

These relationships are recursive, and involve a single inversion of the linearized stiffness matrix. They also involve higher order derivatives of the equivalent load vector F_i, and dealing with low order, simple algebraic expressions simplifies this task greatly. Following the perturbation approach, the solution could be written as:

$$u_i = \bar{u}_i + \lambda\, u_i^{(1)} + \lambda^2/2\, u_i^{(2)} + \lambda^3/6\, u_i^{(3)} + \lambda^4/24\, u_i^{(4)} + \dots, \tag{26}$$

however, the convergence characteristics of this expansion are extremely poor as was shown in [17]. A much better approach is that proposed by Noor *et al.* [18-19] where the perturbation modes are used as assumed modes, following a modal approach as described in Section 4.

6. NUMERICAL EXAMPLES

Two examples will be used to demonstrate the efficiency and accuracy of the modal procedure by comparing its predictions with that of the full finite element model described in section 3. First, the problem of a cantilevered beam with a solid aluminum cross-section treated previously will be considered. The 45° loading angle case was selected for this comparison as it exhibits the strongest coupled nonlinear behavior. The modal approximation was run with an increasing number of modes, from one perturbation mode (i.e. the linear solution of the problem), up to five perturbation modes. The results are summarized in Table 1 which lists the percentage error between the full finite element results and the various modal predictions of the tip twist, tip flatwise and edgewise displacements, root torque, and tip axial force, for 1, 2, and 3 lbs tip load. The single mode approximation did not converge for the 3 lb case.

TABLE 1: COMPARISON BETWEEN THE FULL FINITE ELEMENT FORMULATION AND THE MODAL APPROACH. ERRORS IN [%]. Nc MEANS NO CONVERGENCE OF THE MODAL APPROACH.

	tip load [lbs]	1 Mode [%]	2 Mode [%]	3 Mode [%]	4 Mode [%]	5 Mode [%]
tip	1.00	23.4	-0.47	-0.39	-0.02	0.00
twist	2.00	10.5	-1.93	-1.49	-0.00	0.04
angle	3.00	Nc	-4.41	-3.08	-0.00	0.02
flatwise	1.00	-2.96	-0.36	0.01	-0.00	0.00
displa-	2.00	-14.2	-1.44	0.16	-0.03	-0.00
cement	3.00	Nc	-3.25	0.78	-0.11	-0.00
edgewise	1.00	2.55	5.01	-0.14	0.03	0.00
displa-	2.00	6.31	16.8	-1.81	0.34	0.00
cement	3.00	Nc	29.6	-7.01	1.16	0.03
Root	1.00	100.0	-0.66	-0.47	-0.00	0.00
	2.00	100.0	-2.64	-1.75	-0.05	0.02
Torque	3.00	Nc	-5.87	-3.54	-0.20	0.11
Tip	1.00	100.0	0.00	0.20	0.00	0.00
Axial	2.00	100.0	-0.04	0.82	0.00	-0.01
Force	3.00	Nc	-0.26	1.95	-0.01	-0.02

Table 1 shows that with three perturbation modes only, a good approximation is obtained, and for a five mode approximation the discrepancy is within one tenth of one percent. Furthermore, the same level of accuracy is obtained for both displacements and internal forces as should be expected from a Reissner's Principle based model. This contrasts with displacement based modal approximation for which the accuracy of the internal forces is much poorer than that of the displacements.

The second example deals with the harmonic response of a clamped-clamped beam under a known base excitation. Tseng and Dugundji [24] analysed this problem using the harmonic balance method and also reported experimental results obtained from a shake test. Finite elements in time were used to obtain the periodic solution of the problem, and the results of a full finite element representation of the blade obtained in [10] are compared with that of a modal representation. A total of four modes were used: the first natural vibration mode of the beam in bending, two perturbations modes about this natural bending mode, and the second natural bending mode. The perturbation modes were obtained by using as loading vector the inertia forces associated with the first bending mode, hence, for (22), the first perturbation modes is the mode shape itself. Figure 3 shows the normalized mid-span deflection w versus the normalized excitation frequency ω, defined as

$$\bar{w} = \frac{w}{h}\frac{\omega_1}{\omega_f} \quad , \text{ and } \bar{\omega} = \frac{\omega_f}{\omega_1} \tag{27}$$

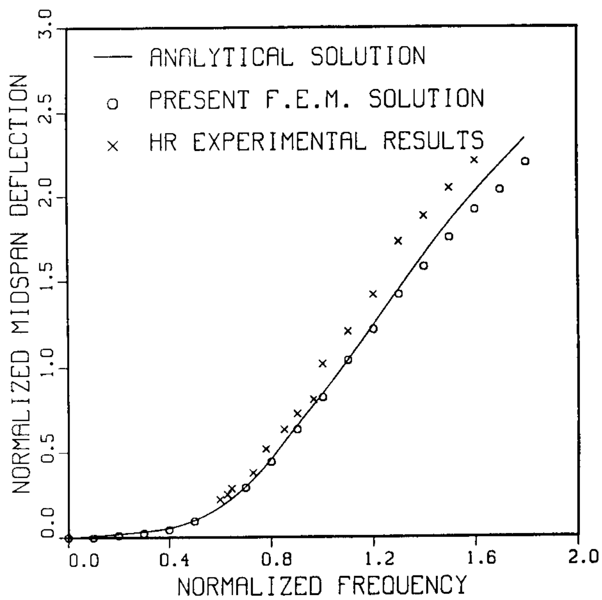

Fig. 3. Harmonic response of a clamped-clamped beam.

where w is the beam's mid-span deflection, h its height, ω_1 its first natural bending frequency, and ω_f the excitation frequency. The experimental results, an analytical solution, the full finite element solution, and the modal approximation are shown in Fig. 3 (the latter two solution superpose on the figure). Good agreement is found among the various solutions, except at large amplitude where the theories slightly underpredict mid-span deflections. The important point however, is the virtually exact correlation between the modal approximation and the full finite element model for this dynamic problem.

7. CONCLUSION

In this paper a formulation was presented for the arbitrarily large displacement and rotation of a naturally curved and twisted composite blade. The formulation is based on Reissner's Principle, and Euler Parameters are used to represent the finite rotations. The resulting governing equations were shown to contain simple algebraic nonlinearities only, of the cubic order. Such a formulation is ideally suited for a modal approximation of the problem since no additional assumption is required to accomodate the modal representation.

Furthermore, the concept of perturbation modes was introduced and shown to yield an extremely efficient and accurate solution procedure. Numerical examples show an excellent agreement between modal representation and full finite element solutions for both static and dynamic problems.

APPENDIX 1

A rotation matrix can be expressed as a function of the Euler Parameters q_0, q_1, q_2, q_3 as follows [20]:

$$T = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}. \tag{A1}$$

However, the four Euler Parameters are related through the normality condition:

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \tag{A2}$$

The curvatures of the reference line can be expressed in terms of these parameters as:

$$\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = 2 \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \begin{bmatrix} q_0' \\ q_1' \\ q_2' \\ q_3' \end{bmatrix} \tag{A3}$$

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