Intrinsic Time Integration Procedures for Rigid Body Dynamics*

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Abstract

The treatment of rotations in rigid body and Cosserat solids dynamics is challenging. In most cases, at some point in the formulation, a parameterization of rotation is introduced and the intrinsic nature of the equations of motions is lost. Typically, this step considerably complicates the form of the equations and increases the order of the nonlinearities. Clearly, it is desirable to bypass parameterization of rotation, leaving the equations of motion in their original, intrinsic form. This has prompted the development of rotationless and intrinsic formulations. This paper focuses on the latter approach. The most famous example of intrinsic formulation is probably Euler's second law for the motion of a rigid body rotating about an inertial point. This equation involves angular velocities solely, with algebraic nonlinearities of the second-order at most. Unfortunately, this intrinsic equation also suffers serious drawbacks: the angular velocity of the body is computed, but not its orientation, the body is "unaware" of its inertial orientation. This paper presents an alternative approach to the problem by proposing discrete statements of the rotation kinematic compatibility equation, which provide solutions for both rotation tensor and angular velocity without relying on a parameterization of rotation. The formulation is also generalized using the motion formalism, leading to very simple discretized equations of motion.

1 Introduction

The equations of motion of a rigid body typically involve the displacement of one reference point of the body and its orientation. The governing equations also involve the velocity and acceleration of the reference point, and the body's angular velocity and acceleration. More generally, beams and shells are Cosserat solids whose kinematic description require both displacement and rotation fields. The geometrically exact equations of motion of beams or shells involve the displacement of one reference point of the cross-section or normal material line, respectively, and the orientation of the corresponding entity. Of course, both spatial and temporal derivatives of these fields are also present in the governing equations of motion.

A crucial difficulty associated with these types of problems is the proper treatment of rotations, because they do not form a linear space. It is therefore not unexpected that many approaches have been proposed to eliminate rotations from the formulations. It is beyond the scope of this paper to provide an exhaustive review of these approaches. Rather, *rotationless* and *intrinsic* formulations are briefly discussed here as they impact the developments presented in the present paper.

Rotationless formulations were developed by García de Jalón *et al.* who introduced the concept of basic coordinates for systems composed rigid bodies; the approach was initially developed for the

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kinematic analysis of planar lower-pair mechanisms [1, 2] and later expanded to deal with spatial mechanisms [3, 4]. García de Jalón *et al.* [5] later showed how natural coordinates evolved from the earlier basic coordinates, and used this new concept to describe multibody systems.

Betsch and his coworkers also developed rotationless formulations for rigid body dynamics [6], beams [7], and plates and shells [8]. The kinematic variables of the problem include the displacement of a reference point and the nine components of the direction cosine matrix. Romero *et al.* [9, 10] have developed similar methodologies for rods.

The main attraction of these formulations is that the resulting governing equations of motion for rigid multibody systems, beams, and plates and shells involve algebraic nonlinearities, of the second degree at most. For these problems, it is remarkable that the midpoint time integration scheme naturally leads to discrete equations that satisfy energy and momentum conservation conditions.

Of course, kinematic constraints must be enforced because the nine components of the direction cosine matrix are not independent. To deal with these constrains, García de Jalón used Maggi's and the null space formulations, whereas Betsch *et al.* [11, 12, 13] developed the discrete null space approach. Clearly, when using rotationless formulations, the problems associated with the treatment of rotation have been eliminated, but the formulation must now deal with an equally thorny problem, that of nonlinear kinematic constraints.

Intrinsic formulations are the result of a different strategy to deal with rotations. In this approach, rotations are used to represent system configuration, but are eliminated from the final equations of motion. The most famous example of an intrinsic equation is probably Euler's second law for the motion of a rigid body rotating about an inertial point. The equation of motion is $\dot{\underline{h}} = \underline{M}$, where $\underline{h}$ and $\underline{M}$ are the angular momentum and externally applied moment vectors, respectively, both computed about the inertial point. The angular momentum is defined as $\underline{h} = \underline{\underline{R}}\,\underline{\underline{\varrho}}^*\underline{\underline{R}}^T\underline{\omega}$, where $\underline{\underline{R}}$ is the rotation tensor that describes the orientation of the body, $\underline{\underline{\varrho}}^*$ are the components of the moment of inertia tensor about the inertial point resolved in the material basis, and $\underline{\omega}$ is the angular velocity vector.

In this form, Euler's second law is not intrinsic because the rotation tensor appears in the equations explicitly. Simple algebraic manipulations, however, allow this law to be recast as $\underline{\underline{\varrho}}^*\dot{\underline{\omega}}^* + \tilde{\omega}^*\underline{\underline{\varrho}}^*\underline{\omega}^* = \underline{M}^*$, where notation $(\cdot)^*$ indicates tensor components resolved in the material basis. This intrinsic form of Euler's second law is well known. Note that references to the rotation tensor have been eliminated and the nonlinear terms are of an algebraic nature, of second-order only.

Intrinsic equations for the dynamics of transversely isotropic beams undergoing extension, bending, and torsion have been proposed by Hegemier and Nair [14]. Hodges and his coworkers [15, 16, 17, 18] have developed solution methodologies for these equations without making any use of displacement or rotation parameters. Hodges [15] concludes that "the main advantage of this formulation appears to be that the nonlinearities are of a lower degree. This is because one need not introduce finite rotation variables unless the loads or boundary conditions depend on orientation." Furthermore, discretizations of these intrinsic equations naturally lead to discrete momentum and energy conservation laws.

Unfortunately, intrinsic equations also suffer serious drawbacks. Indeed, a simple glance at the intrinsic form of Euler's second law reveals that it is written in the *material frame*. Although the equations provide a solution of the problem, the body is "unaware" of its inertial orientation. This would be a problem if the components of the externally applied moment were to be given in the inertial frame; the right-hand side of Euler's equation would then be evaluated as $\underline{M}^* = \underline{\underline{R}}^T\underline{M}$, where $\underline{M}$ are the given inertial components of the externally applied moment. Clearly, most practical problems require the simultaneous determination of both angular velocity vector and rotation tensor. When dealing with multibody systems involving rigid bodies, beams, shells, and kinematic joints subjected to arbitrary, deformation-dependent loading, displacement and rotation variables are indispensable ingredients of the formulation [19, 20].

This paper focuses on a new approach to dealing with rotations. The term "intrinsic formulation" will be expanded to indicate a formulation that does include displacements and rotations, but *rotations are not parameterized*. Euler's second law written in the inertial basis, $\underline{\underline{\varrho}}\,\underline{\dot{\omega}} + \tilde{\omega}\,\underline{\underline{\varrho}}\,\underline{\omega} = \underline{M}$, provides a simple example of this type of formulation. In these equations, the components of all vectors and tensors are resolved in the inertial basis, and hence, $\underline{\underline{\varrho}} = \underline{\underline{R}}\,\underline{\underline{\varrho}}^*\,\underline{\underline{R}}^T$; clearly, explicit reference is made to the rotation tensor.

With the new definition of the term "intrinsic," the inertial statement of Euler's law is intrinsic, but the formulation is incomplete because the rotation tensor and the angular velocity are not independent variables. Rather, they are linked by the rotation kinematic compatibility equations, $\underline{\underline{\dot{R}}}\,\underline{\underline{R}}^T = \tilde{\omega}$, which can be taken as a definition of the angular velocity vector. The coupled solution of the dynamical equations, Euler's law, and of the rotation kinematic compatibility equations provide the dynamic response of the system.

The classical approach to this problem is to parameterize rotations. If $\underline{p}$ is a parameterization of rotation, $\underline{\underline{R}} = \underline{\underline{R}}(\underline{p})$ and $\underline{\omega} = \underline{\underline{H}}(\underline{p})\dot{\underline{p}}$, where $\underline{\underline{H}}(\underline{p})$ is the tangent tensor [19, 20]. The governing equation of the problem becomes $\underline{\underline{R}}(\underline{p})\underline{\underline{\varrho}}^*\underline{\underline{R}}^T(\underline{p})[\underline{\underline{\dot{H}}}(\underline{p})\dot{\underline{p}} + \underline{\underline{H}}(\underline{p})\ddot{\underline{p}}] + \widetilde{\underline{\underline{H}}(\underline{p})\dot{\underline{p}}}\,\underline{\underline{R}}(\underline{p})\underline{\underline{\varrho}}^*\underline{\underline{R}}^T(\underline{p})\underline{\underline{H}}(\underline{p})\dot{\underline{p}} = \underline{M}$. Clearly, the parameterization of rotation complicates the final equations considerably. The governing equations are different when different parameterization are used, and are characterized by higher-order nonlinear terms.

The derivation of intrinsic equations for dynamical systems is rather easy. The use of Hamilton's principle in conjunction with transpositional relationships [21, 22] leads to the desired form of the equations for rigid bodies, beams, and shells [20]. In contrast, the use of Lagrange's formulation requires the selection of a parameterization of rotation from the onset of the formulation, leading to arduous derivations and non-intrinsic equations of motion. The rotation kinematic compatibility equations are easily obtained and simply provide the definition of the angular velocity vector.

Clearly, the highly desirable intrinsic form of the equations of motion for dynamical systems is easy to derive. This paper focuses on a solution strategy that does not require a parameterization of rotations, achieving great simplicity while bypassing the shortcomings associated with earlier approaches. The proposed approach is akin to the Lie group integration schemes that have gained increased attention in recent years. The Runge-Kutta methods on Lie groups developed by Munthe-Kaas [23, 24, 25] and Crouch and Grossman [26] provide the theoretical background for this approach, and Iserles *et al.* [27, 28], Celledoni and Owren [29], Kobilarov *et al.* [30] and Brüls *et al.* [31, 32] modified the methodology for application to rigid body dynamics.

This paper is structured in the following manner. Section 2 describes the proposed discretization of the rotation kinematic compatibility equation, which is used in the following section to derive three intrinsic time integration schemes for rigid body dynamics. Section 4 introduces the motion formalism for a rigid body, proposes a discretization of the motion kinematic compatibility equation and derives three intrinsic time integration schemes for rigid body dynamics. Finally, numerical examples are presented in section 5.

2 The rotation kinematic compatibility equations

The kinematic compatibility conditions between the rotation tensor and the angular velocity vector can be expressed in an intrinsic manner as

$$\tilde{\omega} = \underline{\underline{\dot{R}}}\,\underline{\underline{R}}^T, \tag{1a}$$

$$\tilde{\omega}^* = \underline{\underline{R}}^T\,\underline{\underline{\dot{R}}}, \tag{1b}$$

where $\underline{\omega}$ and $\underline{\omega}^*$ are the components of the angular velocity vector in the inertial and material bases, respectively. The rotation kinematic compatibility equations are intrinsic and provide a

definition of the angular velocity vector, but are difficult to use. Indeed, if the rotation tensor is not parameterized, its nine components must be used, but are linked with six orthonormality constraints. The traditional approach to this problem is to parameterize rotations using a minimum set of coordinates, thereby avoiding the use of constraints, but the intrinsic nature of the equations is then lost.

The next section proposes a discretization of the rotation kinematic compatibility equations that does not rely on a parameterization of rotation.

2.1 Discretization of the rotation kinematic compatibility equation

The discretization process focuses on a typical time step starting and ending at times t_i and t_f, respectively, and the time step size is denoted $h = t_f - t_i$. Subscripts $(\cdot)_i$, $(\cdot)_f$, and $(\cdot)_m$ denote quantities evaluated at times t_i, t_f, and $t_m = (t_f + t_i)/2$, respectively, where t_m is the midpoint of the time interval. Figure 1 illustrates the kinematics of the problem. Two orthonormal bases, denoted $\mathcal{B}_i = (\bar{b}_{i1}, \bar{b}_{i2}, \bar{b}_{i3})$ and $\mathcal{B}_f = (\bar{b}_{f1}, \bar{b}_{f2}, \bar{b}_{f3})$, represent the orientation of the rigid body at times t_i and t_f, respectively; the corresponding rotation tensors are denoted $\underline{\underline{R}}_i$ and $\underline{\underline{R}}_f$, respectively. The incremental rotation tensor that brings basis $\mathcal{B}_i$ to basis $\mathcal{B}_f$ is denoted $\underline{\underline{R}}$, *i.e.*, $\underline{\underline{R}}_f = \underline{\underline{R}}\,\underline{\underline{R}}_i$. Resolving the components of the relative rotation tensor in the material basis yields

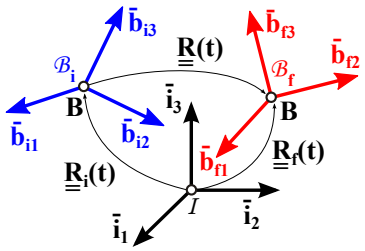

Figure 1: Schematic of the discretization of the rotation tensor at times t_i and t_f.

$$\underline{\underline{R}}_f = \underline{\underline{R}}_i \underline{\underline{R}}^*. \tag{2}$$

Let the relative rotation tensor be defined by rotation ϕ about unit vector $\bar{n}$. The angular velocity can be written in terms of these quantities [20] as $\underline{\omega} = \dot{\phi}\,\bar{n} + \sin\phi\,\dot{\bar{n}} + (1 - \cos\phi)\tilde{n}\dot{\bar{n}}$. In typical numerical applications, the magnitude of body's incremental rotation remains small within each time step, *i.e.*, $\phi_i = 0$ and $\phi_f = \Delta\phi$, where $\Delta\phi \ll 1$. The corresponding angular velocity is approximated as $\underline{\omega} \approx \dot{\phi}\,\bar{n}$. Note that under these conditions, the dependency of the angular velocity vector on the time derivative of unit vector $\bar{n}$ becomes negligible. Because the incremental rotation is small, $\dot{\phi} \approx \Delta\phi/h$, and hence, $h\underline{\omega} \approx \Delta\phi\,\bar{n}$. Finally, since $\Delta\phi$ is a small quantity, it is possible to write

$$h\underline{\omega} \approx p(\Delta\phi)\bar{n}, \tag{3}$$

where p is an arbitrary odd function of angle $\Delta\phi$ such that $\lim_{\Delta\phi\to 0} p(\Delta\phi)/\Delta\phi = 1$. For instance, for the Cayley, Gibbs, Rodrigues parameterization, $p(\phi) = 2\tan\phi/2$ and $p(\phi) = \phi$ for the Cartesian rotation vector; more options can be found in ref. [20].

When expressed in the material basis, eq. (3) becomes $h\underline{\omega}^* \approx p(\Delta\phi)\bar{n}^*$. The right-hand side of this equation is the vectorial parameterization [33, 20], $\underline{p}^* = p(\Delta\phi)\bar{n}^*$, of the relative rotation tensor $\underline{\underline{R}}^*$, *i.e.*, $\underline{\underline{R}}^* = \underline{\underline{R}}^*(\underline{p}^*)$. For very small time step sizes,

$$h\underline{\omega}^* = \underline{p}^* = p(\Delta\phi)\bar{n}^*, \tag{4}$$

and eq. (2) becomes

$$\underline{\underline{R}}_f = \underline{\underline{R}}_i \underline{\underline{R}}^*(h\underline{\omega}^*). \tag{5}$$

This relationship is the "rotation update equation" that enables the update of the rigid body's orientation from time t_i to time t_f. Because this update equation involves the product of two orthogonal matrices, the final orientation, $\underline{\underline{R}}_f$, is guaranteed to be an orthogonal matrix; this observation is the starting point for Lie group integrators. Furthermore, the update equation does not involve a parameterization of rotation, hence, it is an *intrinsic rotation update equation*.

The geometric interpretation of the intrinsic update equation is easily obtained. During one time step, the rotation of the rigid body is characterized by a rotation of magnitude $\Delta\phi$ about a fixed unit vector, $\bar{n}^*$. During the same time step, the angular velocity, $\underline{\omega}^*$, of the body is constant and $h\underline{\omega}^* = \underline{p}^*$, where $\underline{p}^* = p(\Delta\phi)\bar{n}^*$ represents any vectorial parameterization of the relative rotation tensor, $\underline{\underline{R}}^*$, as expressed by eq. (5).

The above developments are based on the assumption that the incremental rotation remains small; *i.e.*, $\Delta\phi \ll 1$. The accuracy of this approximation, however, has not been established. The rotation tensor affords the following multiplicative decomposition, $\underline{\underline{R}}^*(\underline{p}^*) = (\underline{\underline{I}}+\varepsilon\tilde{p}^*/2)(\underline{\underline{I}}-\varepsilon\tilde{p}^*/2)^{-1}$, where $\varepsilon = (2\tan\Delta\phi/2)/\|\underline{p}^*\|$. Introducing this decomposition into the intrinsic rotation update equation yields $\underline{\underline{R}}_f(\underline{\underline{I}} - \varepsilon\tilde{p}^*/2) = \underline{\underline{R}}_i(\underline{\underline{I}} + \varepsilon\tilde{p}^*/2)$, and simple algebraic manipulations then lead to

$$\frac{\underline{\underline{R}}_f - \underline{\underline{R}}_i}{h} = \frac{\underline{\underline{R}}_f + \underline{\underline{R}}_i}{2}\varepsilon\tilde{\omega}^*, \tag{6}$$

where eq. (4) was used to express the vectorial parameterization in terms of the angular velocity.

Equation (6) is easily recognized as a central difference approximation of the rotation kinematic compatibility equation, eq. (1b), written as $\dot{\underline{\underline{R}}} = \underline{\underline{R}}\tilde{\omega}^*$. Indeed, $\underline{\underline{R}}(t_m) \approx (\underline{\underline{R}}_f + \underline{\underline{R}}_i)/2$ and $\dot{\underline{\underline{R}}}(t_m) \approx (\underline{\underline{R}}_f - \underline{\underline{R}}_i)/h$ provide second-order approximations the rotation tensor and its derivative, respectively, at the midpoint of the time step. Next, the angular velocity is selected as $\underline{\omega}^*(t_m) = \underline{\omega}^*_m = (\underline{\omega}^*_f + \underline{\omega}^*_i)/2$, providing once again a second-order, midpoint approximation. Finally, the small angle approximation reveals that $\varepsilon = 1 + \mathcal{O}(\Delta\phi)^2$.

In summary, letting the parameter vector of the relative rotation tensor equal the product of the angular velocity vector by the time step size, as expressed by eq. (4), leads to a second-order approximation of the rotation kinematic equation, see eq. (6). This approximation remains second-order accurate for any vectorial parameterization of rotation since the choice of different generating functions, $p(\Delta\phi)$, only affects the value of parameter ε, which always satisfies condition $\varepsilon = 1 + \mathcal{O}(\Delta\phi)^2$.

The geometric interpretation of the intrinsic update equation now becomes more precise: during a time step, the rigid body rotates at a constant angular velocity, $\underline{\omega}^*_m = (\underline{\omega}^*_f + \underline{\omega}^*_i)/2$, which is parallel to the parameter vector of the relative rotation tensor that describes the incremental rotation of the rigid body, $h\underline{\omega}^*_m = \underline{p}^*$.

An alternative interpretation of eq. (5) is found by recasting it as $\underline{\underline{R}}_i^T\underline{\underline{R}}_f = \underline{\underline{R}}^*(\underline{p}^*)$ leading to $(\underline{\underline{R}}_i^T\underline{\underline{R}}_f) - (\underline{\underline{R}}_i^T\underline{\underline{R}}_f)^T = \underline{\underline{R}}^* - \underline{\underline{R}}^{*T} = 2\zeta_1\ \tilde{p}^*$, where $\zeta_1 = (\sin\phi)/\|\underline{p}^*\|$ [20]. Simple algebraic manipulations then yield

$$\frac{\underline{\underline{R}}_f^T + \underline{\underline{R}}_i^T}{2}\frac{\underline{\underline{R}}_f - \underline{\underline{R}}_i}{h} = \zeta_1\tilde{\omega}^*. \tag{7}$$

Clearly, this corresponds to a second-order accurate approximation of the rotation kinematic equation stated as $\underline{\underline{R}}^T\dot{\underline{\underline{R}}} = \tilde{\omega}^*$. Here again, it is easily verified that $\zeta_1 = 1 + \mathcal{O}(\Delta\phi)^2$. Various discretized forms of the rotation kinematic equation can be obtained from eq. (4), but all are equivalent within a second-order term.

The rotation update equation given by eq. (5) already appears in the work of Simo and Vu-Quoc [34, 35] in the context of geometrically exact beam formulations, and in Simo and Wong [36], as applied to rigid body dynamics. These papers, however, aim at the development of energy and momentum preserving algorithms rather than the development of intrinsic schemes.

3 Rotation of a rigid body

Consider a rigid body free to rotate in three-dimensional space while one of its material points, point **B**, remains at a fixed inertial location. Using Hamilton's principle, the equations of motion

of this problem are found easily

$$\dot{\underline{h}} = \underline{M}, \tag{8a}$$

$$\underline{h} = \underline{\underline{R}}\,\underline{\underline{\varrho}}^*\,\underline{\underline{R}}^T\underline{\omega}, \tag{8b}$$

where $\underline{M}$ is the sum of the moments externally applied to the rigid body with respect to point **B** and $\underline{\underline{\varrho}}^*$ are the components of the mass moment of inertia tensor about the same point, expressed in the material basis. Equation (8a) is Euler's second law, whereas eq. (8b) simply defines the angular momentum vector, $\underline{h}$. The derivation of the equations using Hamilton's principle does not require the use of a particular parameterization of rotation: eqs. (8) are intrinsic and hold for any parameterization of rotation. Of course, system (8) must be complemented with the rotation kinematic compatibility equation to form a complete set.

The following sections develop discretized versions of the dynamical equations, eq. (8), that keep their intrinsic nature, but all use the discrete kinematic compatibility equation, eq. (5).

3.1 Momentum and energy preserving scheme

The following discretization of the governing equations of motion, eqs. (8), is proposed

$$\frac{\underline{h}_f - \underline{h}_i}{h} = \underline{M}_m, \tag{9a}$$

$$\underline{h}_f = \underline{\underline{R}}_f\,\underline{\underline{\varrho}}^*\,\underline{\omega}^*_f. \tag{9b}$$

$$\underline{\underline{R}}_f = \underline{\underline{R}}_i\,\underline{\underline{R}}^*(h\underline{\omega}^*_m). \tag{9c}$$

The first equation is a discretization of Euler's second law based on the central difference scheme. It can also be interpreted as a statement of the principle of impulse and momentum for the rigid body, $\underline{h}_f - \underline{h}_i = \int_{t_i}^{t_f} \underline{M}\ \mathrm{d}t$, where the angular impulse of the externally applied moment has been approximated using the mean value theorem, $\int_{t_i}^{t_f} \underline{M}\ \mathrm{d}t \approx h\underline{M}_m$. The second equation simply defines the angular momentum at time t_f, and finally, the last equation restates the intrinsic update formula, eq. (5).

Equations (9) form a set of discretized, intrinsic equations for the dynamics of a rigid body. The angular velocity and orientation of the body are the unknowns of the problem. Eliminating the unknown angular momentum, $\underline{h}_f$, from eqs. (9a) and (9b) yields a single equation for the components of the angular velocity vector, $\underline{h}_i + h\underline{M}_m = \underline{\underline{R}}_i\underline{\underline{R}}^*(h\underline{\omega}^*_m)\underline{\underline{\varrho}}^*\underline{\omega}^*_f$. The solution of this nonlinear equation yields the angular velocity and the update formula then gives the final orientation of the rigid body. The solution process yields both angular velocity and orientation of the rigid body using intrinsic equations only.

The integration scheme expressed by eqs. (9) presents remarkable properties. First, in the absence of externally applied moments, it satisfies the condition of preservation of angular momentum. This should be expected because eq. (9a) is a discrete statement of the principle of impulse and momentum. Second, the algorithm also satisfies the principle of work and energy. Equation (5) implies $\underline{\underline{R}}_f\underline{\omega}^*_m = \underline{\underline{R}}_i\underline{\omega}^*_m$ and pre-multiplication of eq. (9a), recast as $\underline{\underline{R}}_f\underline{h}^*_f - \underline{\underline{R}}_i\underline{h}^*_i = h\underline{M}_m$, by $\underline{\omega}^{*T}_m\underline{\underline{R}}^T_f = \underline{\omega}^{*T}_m\underline{\underline{R}}^T_i$ yields $\underline{\omega}^{*T}_m(\underline{h}^*_f - \underline{h}^*_i) = h\underline{\omega}^{*T}_m(\underline{\underline{R}}^T_f + \underline{\underline{R}}^T_i)\underline{M}_m/2$. Observing that $\underline{\omega}^{*T}_m(\underline{h}^*_f - \underline{h}^*_i) = (\underline{\omega}^*_f + \underline{\omega}^*_i)\underline{\underline{\varrho}}^*(\underline{\omega}^*_f - \underline{\omega}^*_i)/2 = K_f - K_i$, where K_f and K_i are the kinetic energies of the rigid body at times t_f and t_i, respectively, and that $h\underline{\omega}^{*T}_m(\underline{\underline{R}}^T_f + \underline{\underline{R}}^T_i)\underline{M}_m/2 = \Delta\phi\bar{n}^{*T}(\underline{\underline{R}}^T_f + \underline{\underline{R}}^T_i)\underline{M}_m/2 = \Delta W$ is the incremental work done by the externally applied moment, it follows that $K_f - K_i = \Delta W$, which is a discrete statement of the principle of work and energy. Third, in view of the discrete energy preservation property of the scheme, it is unconditionally stable.

3.2 Momentum preserving energy decaying scheme

The discretization proposed in the previous section presents a desirable feature: it preserves both energy and momentum. Although such schemes perform well for simple rigid body applications, the absence of high-frequency numerical dissipation causes problems when dealing with flexible multibody systems [37]. In fact, it is well established [38, 39] that high-frequency numerical dissipation is an indispensable feature of practical time integration schemes.

It is possible to develop schemes presenting high-frequency numerical dissipation by proving an energy decay inequality [40]. This approach was followed by Bauchau and his co-workers to develop energy decaying schemes for beams [41], elastodynamics [42], multibody systems [43, 44, 45], and plates and shells [46, 47]. These schemes originate from time discontinuous Galerkin approximations of the equations of motion written in the symmetric hyperbolic form. Bottasso and Borri proposed both energy preserving and decaying schemes for beams [48, 49] and multibody systems [50, 51]. Their schemes were cast first within the framework of finite elements in time, then as 2-stage FSAL Runge-Kutta methods. All these schemes, however, were based on specific parameterizations of finite rotations.

Based on the procedure developed by Bauchau [41, 43, 44], the scheme proposed in section 3.1 can be extended to include high-frequency numerical dissipation

$$\frac{\underline{h}_f - \underline{h}_i}{h} = \frac{\underline{M}_f + \underline{M}_i}{2}, \quad \frac{\underline{h}_j - \underline{h}_i}{h} = -\frac{\underline{M}_f - \underline{M}_i}{6}, \tag{10a}$$

$$\underline{h}_f = \underline{\underline{R}}_f \, \underline{\underline{\varrho}}^* \underline{\omega}_f^*, \quad \underline{h}_j = \underline{\underline{R}}_j \, \underline{\underline{\varrho}}^* \underline{\omega}_j^*, \tag{10b}$$

$$\underline{\underline{R}}_f = \underline{\underline{R}}_i \underline{\underline{R}}^* (h \frac{\underline{\omega}_f^* + \underline{\omega}_j^*}{2}), \quad \underline{\underline{R}}_j = \underline{\underline{R}}_i \underline{\underline{R}}^* (-h \frac{\underline{\omega}_f^* - \underline{\omega}_j^*}{6}), \tag{10c}$$

where notation $(\cdot)_j$ indicates the value of the corresponding quantity across the discontinuity at time t_i. Through simple algebraic manipulations, eqs. (10a) and (10b) can be recast as a system of two coupled equations for the unknown angular velocities, $\underline{\omega}_f^*$ and $\underline{\omega}_j^*$. The configuration of the body is then recovered using the intrinsic update equations, eq. (10b).

It is left to the reader to verify that eqs. (10) imply preservation of momentum in the absence of externally applied loads. It can also be verified that the following energy decay inequality is satisfied: $K_f \leq K_i$. More precisely, $K_f = K_i - c^2$, where $c^2 = (\underline{\omega}_j^{*T} - \underline{\omega}_j^{*T}) \underline{\underline{\varrho}}^* (\underline{\omega}_j^* - \underline{\omega}_j^*)/2 \geq 0$. Finally, the energy decay inequality implies the unconditional stability of the scheme.

Unfortunately, the scheme defined by eqs. (10) is computationally expensive because the solution phase requires the computation of the angular velocities at two distinct instants, $\underline{\omega}_f^*$ and $\underline{\omega}_j^*$. This doubling of the number of unknowns will result in an eightfold increase in computational cost. Yet, the scheme is, in general, second-order accurate only, as shown by Bottasso *et al.* [52, 53].

3.3 Schemes possessing high-frequency numerical dissipation

To remedy the situation, schemes developed for classical structural dynamics problems, such as the Hilber-Hughes-Taylor [54] or generalized-α [55] schemes, have also been applied to multibody problems. For linear problems, these schemes achieve high-frequency numerical dissipation while minimizing unwanted low-frequency dissipation. Both methods have been successfully used for both linear and nonlinear problems, although unconditional stability is proved for linear systems only. For instance, Cardona and Géradin [56] used the Hilber-Hughes-Taylor scheme for integrating the equations of motion in mechanism analysis.

To apply the generalized-α scheme, the equations of motion are first recast in second-order form. This is achieved by eliminating the angular momentum from eqs. (8) to find $\underline{\underline{\varrho}} \underline{\dot{\omega}} + \tilde{\omega} \underline{\underline{\varrho}} \underline{\omega} = \underline{M}$. For

this scheme, this equation will be satisfied at discrete time t_f, *i.e.*,

$$\underline{\underline{\varrho}}_f \underline{\dot{\omega}}_f + \tilde{\omega}_f \underline{\underline{\varrho}}_f \underline{\omega}_f = \underline{M}_f. \tag{11}$$

The generalized-α scheme defines a family of time integrators characterized by the following approximations for the incremental rotation, $\underline{\Delta\psi}$, and angular velocity of the body,

$$\underline{\Delta\psi} = h\underline{\omega}_i + h^2 \left[(\frac{1}{2} - \beta)\underline{a}_i + \beta \underline{a}_f \right], \tag{12a}$$

$$\underline{\omega}_f = \underline{\omega}_i + h \left[(1 - \gamma)\underline{a}_i + \gamma \underline{a}_f \right], \tag{12b}$$

where β and γ are two parameters that will be selected to achieve desirable stability and accuracy characteristics for the scheme. These equations use algorithmic accelerations, $\underline{a}_f$ and $\underline{a}_i$, which are related to the actual accelerations of the system through the following recurrence relationship,

$$(1 - \alpha_m)\underline{a}_f + \alpha_m \underline{a}_i = (1 - \alpha_f)\underline{\dot{\omega}}_f + \alpha_f \underline{\dot{\omega}}_i. \tag{13}$$

where α_m and α_f are two additional parameters that will be selected to achieve desirable stability and accuracy characteristics for the scheme. The generalized-α scheme used here is that presented by Arnold and Brüls [57] rather than the original scheme presented by Chung and Hulbert [55]. The final equation of the scheme is the intrinsic rotation update, eq. (5), written here as $\underline{\underline{R}}_f = \underline{\underline{R}}(\underline{\Delta\psi})\underline{\underline{R}}_i$.

4 Motion of a rigid body

Consider now the general motion of a rigid body in three-dimensional space. Using Hamilton's principle and the motion formalism [20], the equations of motion of the problem are found as

$$\underline{\dot{\mathcal{P}}} = \underline{\mathcal{A}}, \tag{14a}$$

$$\underline{\mathcal{P}} = \underline{\underline{\mathcal{C}}}^{-T} \underline{\underline{\mathcal{M}}}^*_B \underline{\underline{\mathcal{C}}}^{-1} \underline{\mathcal{V}}. \tag{14b}$$

Equation (14a) combines Euler's first and second laws for the rigid body, whereas eq. (14b) simply defines the momentum array, $\underline{\mathcal{P}}^T = \left\{ \underline{p}^T, \underline{h}^T \right\}$, where $\underline{p}$ and $\underline{h}$ denote the linear and angular momentum vectors, respectively. The 6×6 mass matrix of the rigid body resolved in the material frame is

$$\underline{\underline{\mathcal{M}}}^*_B = \begin{bmatrix} m\underline{\underline{I}} & m\tilde{\eta}^{*T} \\ m\tilde{\eta}^* & \underline{\underline{\varrho}}^* \end{bmatrix}, \tag{15}$$

where m is the total mass of the body, $\underline{\eta}$ the relative position vector of the body's center of mass with respect to reference point **B**, and $\underline{\underline{\varrho}}^*$ the mass moment of inertia tensor with respect to the same point. Array $\underline{\mathcal{A}}^T = \left\{ \underline{F}^T, \underline{M}^T \right\}$ stores the externally applied forces and moments, denoted $\underline{F}$ and $\underline{M}$, respectively. Finally, the motion tensor, $\underline{\underline{\mathcal{C}}}$, is defined as

$$\underline{\underline{\mathcal{C}}} = \begin{bmatrix} \underline{\underline{R}} & \tilde{u}\underline{\underline{R}} \\ \underline{\underline{0}} & \underline{\underline{R}} \end{bmatrix}, \tag{16}$$

where $\underline{\underline{R}}$ is the rotation tensor that determines the orientation of the rigid body and $\underline{u}$ the displacement vector of reference point **B**. The velocity vector, $\underline{\mathcal{V}}^T = \left\{ \underline{v}^T, \underline{\omega}^T \right\}$, stores the velocity vector of the inertial reference point, $\underline{v}$, and the angular velocity, $\underline{\omega}$, of the body. Note the parallelism between eqs. (14) for the motion of a rigid body and eqs. (8) for the rotation of the body about a fixed inertial point.

Equations (14) are intrinsic equations. The configuration of the rigid body is defined by motion tensor $\underline{\underline{\mathcal{C}}}$, and its velocity by vector $\underline{\mathcal{V}}$. The derivation of the equations using Hamilton's principle does not require the use of a particular parameterization of motion, and eqs. (14) hold for any parameterization of motion.

Equations (14) are intrinsic but incomplete; the kinematic compatibility conditions between the motion tensor and the velocity vector can be expressed in an intrinsic manner as

$$\tilde{\mathcal{V}} = \dot{\underline{\underline{\mathcal{C}}}}\,\underline{\underline{\mathcal{C}}}^{-1}, \tag{17a}$$

$$\tilde{\mathcal{V}}^* = \underline{\underline{\mathcal{C}}}^{-1}\dot{\underline{\underline{\mathcal{C}}}}, \tag{17b}$$

where $\underline{\mathcal{V}}$ and $\underline{\mathcal{V}}^*$ are the components of the velocity vector in the inertial and material frames, respectively. The motion kinematic compatibility equations are intrinsic and provide a definition of the velocity vector [20], but are difficult to use if the motion tensor is not parameterized.

As was the case for the rotating rigid body, eqs. (14) can be recast in the following form, $\underline{\underline{\mathcal{M}}}^*_B\dot{\underline{\mathcal{V}}}^* - \tilde{\mathcal{V}}^{*T}\underline{\underline{\mathcal{M}}}^*_B\underline{\mathcal{V}}^* = \underline{\mathcal{A}}^*$. This intrinsic equation, from which the motion tensor has been eliminated, generalizes Euler's equation for the rotational behavior of the rigid body to the problem of general motion. The intrinsic equations of motion in the inertial frame are $\underline{\underline{\mathcal{M}}}_O\dot{\underline{\mathcal{V}}} - \tilde{\mathcal{V}}^T\underline{\underline{\mathcal{M}}}_O\underline{\mathcal{V}} = \underline{\mathcal{A}}$, where $\underline{\underline{\mathcal{M}}}_O = \underline{\underline{\mathcal{C}}}^{-T}\underline{\underline{\mathcal{M}}}^*_B\underline{\underline{\mathcal{C}}}^{-1}$ is the body mass matrix resolved in the inertial frame. The classical approach to the solution of the problem is to introduce a parameterization of motion such that $\underline{\underline{\mathcal{C}}} = \underline{\underline{\mathcal{C}}}(\underline{\mathcal{Q}})$ and $\underline{\mathcal{V}} = \underline{\underline{\mathcal{H}}}(\underline{\mathcal{Q}})\dot{\underline{\mathcal{Q}}}$, where $\underline{\mathcal{Q}}$ is the vectorial parameterization of motion and $\underline{\underline{\mathcal{H}}}$ the tangent tensor. This leads to a set of twelve equations for the twelve unknowns, the components vectors $\underline{\mathcal{P}}$ and $\underline{\mathcal{Q}}$; of course, the intrinsic nature of the equations has been lost.

4.1 Discretization of the motion kinematic compatibility equation

Figure 2 illustrates the kinematics of the problem. Two frames, denoted $\mathcal{F}_i = \left[\mathbf{B_i}, \mathcal{B}_i = (\bar{b}_{i1}, \bar{b}_{i2}, \bar{b}_{i3})\right]$ and $\mathcal{F}_f = \left[\mathbf{B_f}, \mathcal{B}_f = (\bar{b}_{f1}, \bar{b}_{f2}, \bar{b}_{f3})\right]$, represent the configuration of the rigid body at times t_i and t_f, respectively, the corresponding motion tensors are denoted $\underline{\underline{\mathcal{C}}}_i$ and $\underline{\underline{\mathcal{C}}}_f$, respectively. The incremental motion tensor that brings frame $\mathcal{F}_i$ to frame $\mathcal{F}_f$ is denoted $\underline{\underline{\mathcal{C}}}$, *i.e.*, $\underline{\underline{\mathcal{C}}}_f = \underline{\underline{\mathcal{C}}}\,\underline{\underline{\mathcal{C}}}_i$. Resolving the components of the relative motion tensor in the material frame yields

$$\underline{\underline{\mathcal{C}}}_f = \underline{\underline{\mathcal{C}}}_i\underline{\underline{\mathcal{C}}}^*. \tag{18}$$

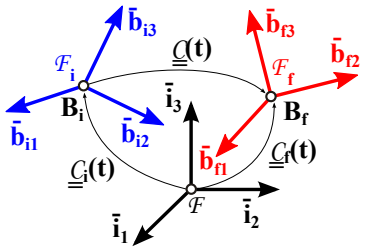

Figure 2: Schematic of the discretization of the motion tensor at times t_i and t_f.

A reasoning similar to that presented in section 2.1 indicates that for very small time step sizes,

$$h\underline{\mathcal{V}}^* = \underline{\mathcal{Q}}^*, \tag{19}$$

where $\underline{\mathcal{Q}}^*$ is a vectorial parameterization of motion [58, 20] and eq. (18) becomes

$$\underline{\underline{\mathcal{C}}}_f = \underline{\underline{\mathcal{C}}}_i\underline{\underline{\mathcal{C}}}^*(h\underline{\mathcal{V}}^*). \tag{20}$$

This relationship is the "motion update equation" that enables the update of the rigid body's configuration from time t_i to time t_f. Because this update equation involves the product of two motion tensors, the final configuration, $\underline{\underline{\mathcal{C}}}_f$, is guaranteed to be a motion tensor. Furthermore, the update equation does not involve a parameterization of motion, hence, it is an *intrinsic motion update equation.*

The accuracy of this approximation has not been established. The motion tensor affords the following multiplicative decomposition [20], $\underline{\underline{\mathcal{C}}}^*(\underline{\mathcal{Q}}^*) = (\underline{\underline{\mathcal{I}}} + \underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon)\underline{\mathcal{Q}}^*/2)(\underline{\underline{\mathcal{I}}} - \underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon)\underline{\mathcal{Q}}^*/2)^{-1}$. Introducing this decomposition into the intrinsic motion update equation yields $\underline{\underline{\mathcal{C}}}_f(\underline{\underline{\mathcal{I}}} - \underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon)\underline{\mathcal{Q}}^*/2) = \underline{\underline{\mathcal{C}}}_i(\underline{\underline{\mathcal{I}}} + \underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon)\underline{\mathcal{Q}}^*/2)$, and simple algebraic manipulations then lead to

$$\frac{\underline{\underline{\mathcal{C}}}_f - \underline{\underline{\mathcal{C}}}_i}{h} = \frac{\underline{\underline{\mathcal{C}}}_f + \underline{\underline{\mathcal{C}}}_i}{2}\underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon)\tilde{\mathcal{V}}^*_m, \tag{21}$$

where eq. (19) was used to express the vectorial parameterization in terms of the velocity, and $\underline{\mathcal{V}}^*_m = (\underline{\mathcal{V}}^*_f + \underline{\mathcal{V}}^*_i)/2$. Equation (21) is easily recognized as a central difference approximation of the rotation kinematic compatibility equation, eq. (17b), written as $\dot{\underline{\underline{\mathcal{C}}}} = \underline{\underline{\mathcal{C}}}\,\tilde{\mathcal{V}}^*$. Tedious algebra reveals that $\varepsilon = 1 + \mathcal{O}(\Delta\phi)^2$ and $\bar{\varepsilon} = \mathcal{O}(\Delta\phi\Delta d)$, where d is the intrinsic displacement of the rigid body. Clearly, $\underline{\underline{\mathcal{Z}}}(\bar{\varepsilon},\varepsilon) = \underline{\underline{\mathcal{I}}} + \mathcal{O}(h)^2$, leading to second-order accuracy.

4.2 Discretization of the equations of motion

Section 3 has presented three different algorithms for the solution of the rotational equations. Given the close similarity between the governing equations for rotation and motion, see eqs. (8) and (14), respectively, and the close similarity between the intrinsic rotation and motion update equations, see eqs. (5) and (20), respectively, the same three algorithms can be developed for the motion of a rigid body. The discretizations are given below and can be obtained by following the procedures outlined in section 3.

By analogy with eqs. (9), the following discretization is proposed

$$\frac{\underline{\mathcal{P}}_f - \underline{\mathcal{P}}_i}{h} = \underline{\mathcal{A}}_m, \tag{22a}$$

$$\underline{\mathcal{P}}_f = \underline{\underline{\mathcal{C}}}_f^{-T}\,\underline{\underline{\mathcal{M}}}^*_B\underline{\mathcal{V}}^*_f, \tag{22b}$$

$$\underline{\underline{\mathcal{C}}}_f = \underline{\underline{\mathcal{C}}}_i\underline{\underline{\mathcal{C}}}^*(h\underline{\mathcal{V}}^*_m). \tag{22c}$$

It is left to the reader to verify that in the absence of externally applied loads, the scheme is both momentum and energy preserving.

By analogy with eqs. (10), an energy decaying scheme for the motion of a rigid body is derived easily

$$\frac{\underline{\mathcal{P}}_f - \underline{\mathcal{P}}_i}{h} = \frac{\underline{\mathcal{A}}_f + \underline{\mathcal{A}}_i}{2}, \quad \frac{\underline{\mathcal{P}}_j - \underline{\mathcal{P}}_i}{h} = -\frac{\underline{\mathcal{A}}_f - \underline{\mathcal{A}}_i}{6}, \tag{23a}$$

$$\underline{\mathcal{P}}_f = \underline{\underline{\mathcal{C}}}_f^{-T}\,\underline{\underline{\mathcal{M}}}^*_B\underline{\mathcal{V}}^*_f, \quad \underline{\mathcal{P}}_j = \underline{\underline{\mathcal{C}}}_j^{-T}\,\underline{\underline{\mathcal{M}}}^*_B\underline{\mathcal{V}}^*_j, \tag{23b}$$

$$\underline{\underline{\mathcal{C}}}_f = \underline{\underline{\mathcal{C}}}_i\underline{\underline{\mathcal{C}}}^*(h\frac{\underline{\mathcal{V}}^*_f + \underline{\mathcal{V}}^*_j}{2}), \quad \underline{\underline{\mathcal{C}}}_j = \underline{\underline{\mathcal{C}}}_i\underline{\underline{\mathcal{C}}}^*(-h\frac{\underline{\mathcal{V}}^*_f - \underline{\mathcal{V}}^*_j}{6}), \tag{23c}$$

Here again, the energy decay property of the scheme is easily proved.

Finally, schemes based on the generalized-α scheme can also be developed. The equations of motion are first recast in second-order form. This is achieved by eliminating the angular momentum from eqs. (14) to find $\underline{\underline{\mathcal{M}}}_O\dot{\underline{\mathcal{V}}} - \tilde{\mathcal{V}}^T\underline{\underline{\mathcal{M}}}_O\underline{\mathcal{V}} = \underline{\mathcal{A}}$. For this scheme, this equation will be satisfied at discrete time t_f, *i.e.*,

$$\underline{\underline{\mathcal{M}}}_{Of}\dot{\underline{\mathcal{V}}}_f - \tilde{\mathcal{V}}^T_f\underline{\underline{\mathcal{M}}}_{Of}\underline{\mathcal{V}}_f = \underline{\mathcal{A}}_f. \tag{24}$$

The incremental motion, $\underline{\Delta\mathcal{U}}$, and velocity of the body are approximated in the following manner

$$\underline{\Delta\mathcal{U}} = h\underline{\mathcal{V}}_i + h^2\left[(\frac{1}{2} - \beta)\underline{a}_i + \beta\underline{a}_f\right], \tag{25a}$$

$$\underline{\mathcal{V}}_f = \underline{\mathcal{V}}_i + h\left[(1 - \gamma)\underline{a}_i + \gamma\underline{a}_f\right]. \tag{25b}$$

These equations use algorithmic accelerations, $\underline{a}_f$ and $\underline{a}_i$, which are related to the actual accelerations of the system through the following recurrence relationship,

$$(1-\alpha_m)\underline{a}_f + \alpha_m \underline{a}_i = (1-\alpha_f)\dot{\underline{\mathcal{V}}}_f + \alpha_f \dot{\underline{\mathcal{V}}}_i. \tag{26}$$

The final equation of the scheme is the intrinsic motion update, eq. (20), written here as $\underline{\underline{\mathcal{C}}}_f = \underline{\underline{\mathcal{C}}}(\underline{\Delta\mathcal{U}})\underline{\underline{\mathcal{C}}}_i$.

5 Numerical examples

The algorithms presented in this paper deal with the rotation and general motion of a rigid body. A cursory look at the formulation reveals that the rotational problem is a subset of the general motion problem. Consequently, the numerical examples presented in this section focus on the general motion of a rigid body.

In general, the equations of motion for a rigid body afford no closed form solution. It is easy, however, to construct an exact solution by solving an inverse dynamics problem. First, the motion of the rigid body is selected; the displacement of its reference point is $u_1 = \sin\omega t$, $u_2 = \cos\omega t - 1$, and $u_3 = 0.5t + \sin 2\omega t$, where $\omega = 2$ rad/s and its rotation is defined by Euler parameters, $e_0 = C_\phi$, $e_1 = S_\phi S_\theta C_\psi$, $e_2 = S_\phi S_\theta S_\psi$, and $e_3 = S_\phi C_\theta$, where angles ϕ, θ, and ψ are given functions of time, $\phi = 2t$, $\theta = t + 2\sin t$, and $\psi = 3\cos t - 3$. Using the properties of Euler motion parameters [20], the components of the velocity and acceleration vectors are then found easily. Finally, the externally applied forces and moments that generate the selected motion of the rigid body are calculated from the equations of motion.

The inertial properties of the rigid body are as follows: the total mass of the body is 6.0 kg, and the components of the position vector of its center of mass and of its mass moments of inertia tensor, both with respect to the reference point, are $\underline{\eta}^{*T} = \{0.25, -0.03, 0.12\}$ m and $\underline{\underline{\varrho}}^* = [1.25, 0.57, -0.23; 0.57, 1.56, 0.34; -0.23, 0.34, 2.54]$ kg·m^2, respectively.

Approximate solutions of the problem were computed for 2 s using time step sizes $h = 4$, 2, 1, 0.5, 0.25, 0.125, 0.0625, and 0.03125 ms, and predictions were compared with the exact solution. The displacement error measure was selected as $e_D = \|\underline{u}_a - \underline{u}_e\|/\|\underline{u}_e\|$, where $\underline{u}_a$ and $\underline{u}_e$ are the approximate and exact displacement vectors at 2 s, respectively. The rotation error measure was selected as $e_R = \|\underline{\underline{R}}_a - \underline{\underline{R}}_e\|$, where $\underline{\underline{R}}_a$ and $\underline{\underline{R}}_e$ are the approximate and exact rotation tensors at 2 s, respectively. For the various examples, the incremental motion tensor, $\underline{\underline{\mathcal{C}}}^*(h\underline{\mathcal{V}}^*)$, was evaluated by treating the incremental motion vector, $h\underline{\mathcal{V}}^*$, as the exponential map of motion, the Euler-Rodrigues, the Cayley-Gibbs-Rodrigues, and the Wiener-Milenković motion parameters; the corresponding error measures are shown in figs. 3 to 8 using symbols ▼, ♦, ■, and ▲, respectively.

First, approximate solutions of the kinematic compatibility equations, eqs. (20), were obtained. Figures 3 and 4 show the error measures versus time step size, on a logarithmic plot. As predicted, the four approximations are second-order accurate.

Next, approximate solutions of the problem were obtained using the energy/momentum preserving scheme, eqs. (22). Figures 5 and 6 show the error measures versus time step size, on a logarithmic plot. For the particular example presented here, the Cayley-Gibbs-Rodrigues motion parameters provide more accurate predictions. This observation is, however, fortuitous, as other parameterizations provide better accuracy for other motion profiles. Suffice to say all predictions are second-order accurate.

Finally, approximate solutions of the problem were obtained using the generalized-α scheme, eqs. (24), using $\rho_\infty = 1$. Figure 7 and 8 show the error measures versus time step size, on a logarithmic plot. As predicted, the approximations are second-order accurate in all four cases. The momentum/energy preserving method appears to yield more accurate predictions than the generalized-α method for the particular example discussed here.

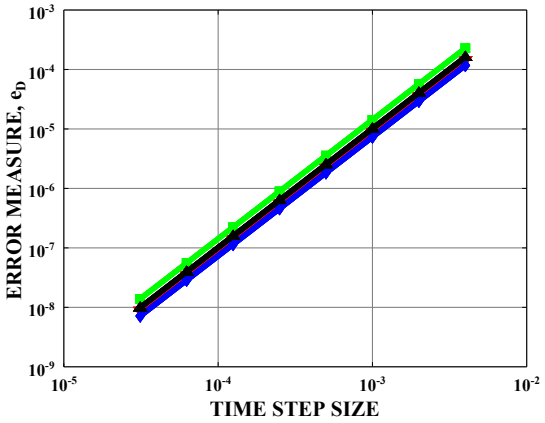

Figure 3: The displacement error measure, e_D, versus time step size for the solution of the motion compatibility equation.

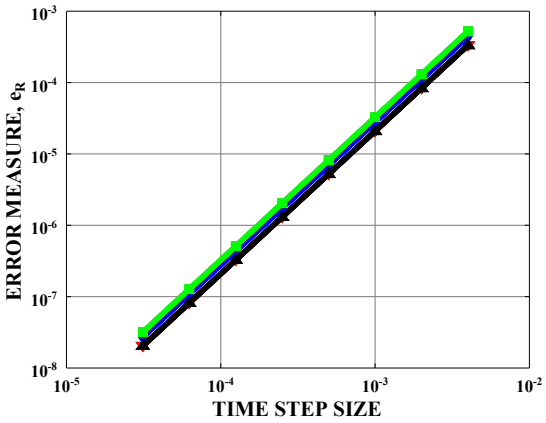

Figure 4: The rotation error measure, e_R, versus time step size for the solution of the motion compatibility equation.

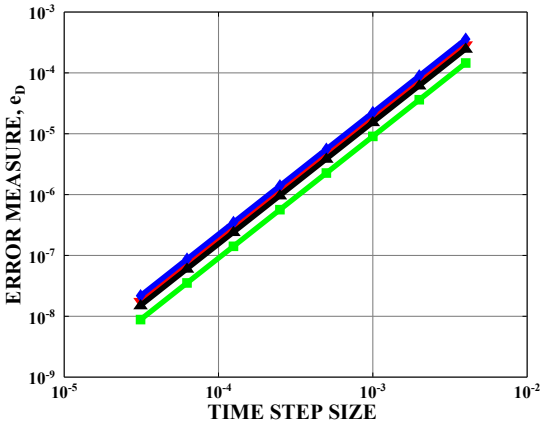

Figure 5: The displacement error measure, e_D, versus time step size for the energy/momentum preserving scheme.

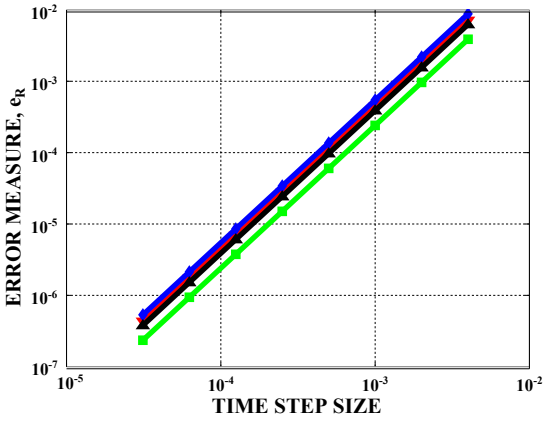

Figure 6: The rotation error measure, e_R, versus time step size for the energy/momentum preserving scheme.

To asses the computational efficiency of the proposed intrinsic formulation, a simple operation count was performed for this rigid body dynamics problem; each operation consists of a multiplication followed by an addition. The total number of operations required to evaluate the Jacobian of the linearized equations of motion for the classical, parameterized formulation is 594. For the proposed intrinsic formulation, the corresponding number is 270, *i.e.*, 45% only of the computation effort required for the classical approach. The reduction of the computational burden stems from the fact that the tangent tensor and its time derivatives are not required in the intrinsic formulation.

6 Conclusions

This paper has presented families of intrinsic time integration schemes for the simulation of rigid body dynamics. The proposed approach is based on discrete approximations of the rotation kinematic compatibility equation. The resulting schemes are called "intrinsic" because they provide solutions for the velocity and orientation of the body without relying on parameterizations of rotation. This approach leads to simple discrete equations presenting low-order algebraic nonlinearities. Furthermore, it eliminates the shortcomings of past intrinsic formulations, which do not solve for the body's orientation. The combination of the proposed discrete rotation kinematic compatibility

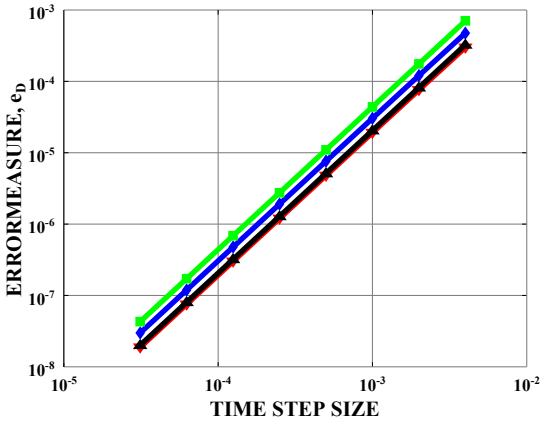

Figure 7: The displacement error measure, e_D, versus time step size for the generalized-α scheme.

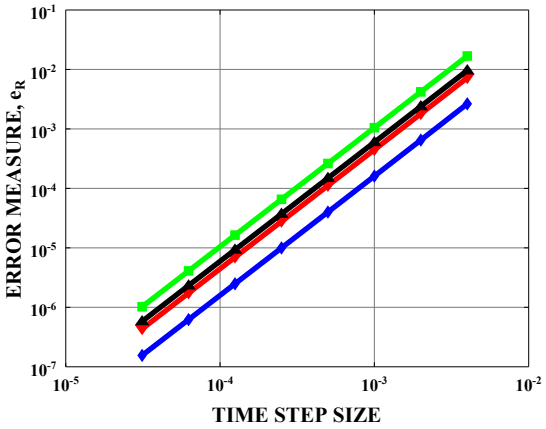

Figure 8: The rotation error measure, e_R, versus time step size for the generalized-α scheme.

equation with discrete forms of the dynamic equations of motion leads to different time integration schemes. This modular approach to the problem enables the development of several families of intrinsic time integration schemes presenting desirable features such as energy/momentum preservation or high-frequency numerical dissipation.

Discrete approximations of the motion kinematic compatibility equation were also presented that generalize the proposed approach to the general motion of the body using the motion formalism. The same modular approach used for the rotation problem then leads to novel integration schemes for the general motion of rigid bodies. Here again, these intrinsic schemes can be tailored to meet specific requirements. The proposed intrinsic schemes have been proved to be second-order accurate for any parameterization of rotation or motion. Because they are formulated in the inertial frame, their extension to Cosserat solids dynamics is possible and will be presented in upcoming papers.

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