Manipulation of Motion via Dual Entities*

Shilei Han† and Olivier A. Bauchau‡
†University of Michigan-Shanghai Jiao Tong University Joint Institute
Shanghai, China 200240
‡Department of Aerospace Engineering, University of Maryland
College Park, Maryland 20742

Abstract

The manipulation of motion and ancillary operations are important tasks in kinematics, robotics, and rigid and flexible multibody dynamics. Motion can be described in purely geometric terms, based on Chasles' theorem. Representations and parameterizations of motion are also available, such as Euler motion parameters and the vectorial parameterization, respectively. Typical operations to be performed on motion involve the selection of local or global parameterizations and the derivation of the associated expressions for the motion tensor, velocity or curvature vector, composition of motions, and tangent tensors. Many of these tasks involve arduous, error-prone algebra. The use of dual entities has been shown to ease the manipulation of motion, yet this concept has received little attention outside of the fields of kinematics and robotics. This paper presents a comprehensive treatment of the topic using a notation that eliminates the bookkeeping parameter typically used in dual number algebra, thereby recasting all operations within the framework of linear algebra and streamlining the process. The manipulation of geometric entities is recast within this formalism, paving the way for the manipulation of motion. All developments are presented within the framework of dual numbers directly; the principle of transference is never invoked: the manipulation of rotation is a particular case of that of motion, as should be. The problem of interpolation of motion, a thorny issue in finite element applications, is also addressed.

1 Introduction

Dual numbers were first introduced in the 19th century by Clifford [1]. Typically, they are written as $\hat{a} = a + \epsilon b$, where a and b are referred to as the primal and dual parts, respectively, and parameter ϵ is such that $\epsilon^n = 0$ for $n \geq 2$. The parallel with complex numbers is evident; a complex number is defined as $\hat{a} = a + ib$, where a and b are referred to as the real and imaginary parts of the complex number, respectively, and parameter i is such that $i^2 = -1$.

Parameters ϵ and i can be thought of as "bookkeeping parameters" that follow special rules, $\epsilon^n = 0$ for $n \geq 2$ and $i^2 = -1$, respectively. A more complicated example is provided by quaternions, which are written as $\hat{a} = a + ib + jc + kd$, where bookkeeping parameters i, j, and k obey the following non-commutative rules: $i^2 = j^2 = k^2 = ijk = -1$.

Complex algebra finds applications in many branches of engineering. On the other hand, the use of dual numbers is restricted to kinematics and that of quaternions to the description of rotation. This is due to the fact that the special rules that govern dual numbers and quaternions are "hard

* *Nonlinear Dynamics* **85**, pp 509–524, 2016

wired" to implement the very specific operations found in kinematics and rotations, respectively. Clearly, non-commutative rule $ijk = -1$ executes the cross-product operation.

Shortly after the development of dual numbers, application to the description of rigid-body kinematics was proposed by Kotelnikov and Study [2]. Kotelnikov proved the principle of transference, which in the words of Rooney [3, 4] states that "All valid laws and formulæ relating to a system of intersecting unit line vectors (and hence, involving real variables) are equally valid when applied to an equivalent system of skew vectors, if each variable a, in the original formula is replaced by the corresponding dual variable $\hat{a} = a + \epsilon b$."

Application of dual number to kinematics is now well established, see Yang and Freudenstein [5], Dimentberg [6], or the textbooks of Bottema and Roth [7] and McCarthy [8]. The geometric interpretation of the rather abstract concept of dual numbers is described by Angeles [9] and Pennestrì and Stefanelli [10] have explored the associated numerical algorithms. Their application to dynamics has been explored by Keler [11] and Brodsky and Shoham [12, 13]. A comprehensive review of the application of dual numbers to various fields is given by Fischer [14].

Despite the efficient and elegant manner by which dual numbers deal with rigid-body motion, their use has remained limited to the field of kinematics. Although rigid-body motion is a key concept in rigid and flexible multibody dynamics, dual numbers are rarely mentioned in these fields. Yet, the implementation of rigid multibody formulations requires extensive manipulation of motion: the problem statement requires the description of the position vectors and orientation tensors of the various bodies, composition of motion is an inherent part of time integration schemes [15, 16], and linearization of the equations of motion calls for local or global parameterizations and for the evaluation of tangent tensors. The same kinematic operations are found in the development of beam [17, 18, 19] or shell models [20, 21] for flexible multibody systems.

A stumbling block in the manipulation of motion is its parameterization, which is arduous and often arcane. For instance, Euler motion parameters, a well-known representation of motion, are related to the dual quaternions used in kinematics [5]. Reviews of motion parameterization techniques have been presented by Borri *et al.* [22], and Bauchau and Choi [23]. In the finite element implementation of beam and shell models, motion must be interpolated, a process fraught with theoretical [24] and numerical [25] difficulties.

While the developments presented in this paper are based on the concept of dual numbers, a novel notation is adopted. Rather than using the traditional bookkeeping parameter ϵ, dual numbers are recast in a matrix formalism. Dual scalars, vectors, and matrices now obey the common rules of linear algebra and the bookkeeping parameter is eliminated. The advantage of this approach is that operations on rotation and motion follow identical patterns, rotation becomes a particular case of motion, as should be; the principle of transference is embedded in the notation. This contrasts with the traditional approach that develops formulæ for rotation, which are then generalized to motion by using the principle of transference.

The goals of this paper are as follows. First, dual numbers are recast within the formalism of linear algebra, see section 2, eliminating the need for bookkeeping parameters. Second, the geometric interpretation of the scalar, vector and tensor products of dual vectors is described in section 2.2. Equipped with these tools, the paper presents all the formulæ needed to manipulate motion based on its geometric description, on Euler parameters, and on the vectorial parameterization, in sections 3, 4, and 5, respectively. In each case, formulæ are presented for the motion tensor, the velocity vector, the composition of motion, and for the tangent tensor. Finally, the interpolation of motion is addressed in section 6.

Throughout the paper, all developments are presented within the framework of dual numbers directly; the principle of transference in never invoked. While no attempt is made to prove this principle, the paper shows that the manipulation of rotation is a particular case of that of motion, as should be.

2 Notational conventions

The classical notation for the vector product, $\underline{c}$, of two vectors, $\underline{a}$ and $\underline{b}$, is $\underline{c} = \underline{a} \times \underline{b}$. It has become common practice to associate skew-symmetric matrix $\tilde{a}$ with the components of vector $\underline{a}^T = \{a_1, a_2, a_3\}$,

$$\tilde{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{1}$$

The vector product operation is now expressed as $\underline{c} = \tilde{a}\underline{b}$, where the standard rules of linear algebra apply. The cross-product operation has been "hard wired" in the structure of skew-symmetric matrix $\tilde{a}$. The inverse operation that extract the components of the vector from the skew symmetric matrix is noted $\underline{a} = \text{axial}(\tilde{a})$.

A quaternion [26] is defined as an array of four numbers $\hat{a}^T = \{a_0, a_1, a_2, a_3\}$, where a_0 is the scalar part of the quaternion and $\underline{a}^T = \{a_1, a_2, a_3\}$ its vector part. The classical notation is $\hat{a} = a_0 + ia_1 + ja_2 + ka_3$. To evaluate the product of two quaternions, $\hat{c} = \hat{a}\hat{b}$, the non-commutative rules of quaternion algebra apply: $i^2 = j^2 = k^2 = ijk = -1$. The same operation is performed more conveniently by associating the following 4×4 matrices [27] with the components of a quaternion

$$\underline{\underline{A}}(\hat{a}) = \begin{bmatrix} a_0 & -\underline{a}^T \\ \underline{a} & a_0\underline{\underline{I}} + \tilde{a} \end{bmatrix}, \quad \underline{\underline{B}}(\hat{a}) = \begin{bmatrix} a_0 & -\underline{a}^T \\ \underline{a} & a_0\underline{\underline{I}} - \tilde{a} \end{bmatrix}. \tag{2}$$

It is left to the reader to verify that quaternion multiplication, $\hat{c} = \hat{a}\hat{b}$, is now recast in terms of linear algebra as $\hat{c} = \underline{\underline{A}}(\hat{a})\hat{b}$. The rules associated with bookkeeping parameters i, j, and k have been "hard wired" in the structure of matrix $\underline{\underline{A}}(\hat{a})$. The introduction of 4×4 matrix $\underline{\underline{B}}(\hat{a})$ is useful to perform other quaternions operations.

The classical notation for dual scalars is $\hat{\alpha} = \alpha^\flat + \epsilon\alpha^\sharp$, where $\alpha^\flat$ and $\alpha^\sharp$ are the primal and dual parts of the dual scalar. Bookkeeping parameter ϵ is such that $\epsilon^n = 0$ for $n \geq 2$. The product of two dual scalars now becomes $\hat{\alpha}\hat{\beta} = (\alpha^\flat + \epsilon\alpha^\sharp)(\beta^\flat + \epsilon\beta^\sharp) = \alpha^\flat\beta^\flat + \epsilon(\alpha^\flat\beta^\sharp + \alpha^\sharp\beta^\flat) + \epsilon^2\alpha^\sharp\beta^\sharp = \alpha^\flat\beta^\flat + \epsilon(\alpha^\flat\beta^\sharp + \alpha^\sharp\beta^\flat)$, where the last equality follows from the rules associated with bookkeeping parameter ϵ. Following the examples of the previous paragraphs, the following 2×2 matrix is introduced

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha^\flat & \alpha^\sharp \\ 0 & \alpha^\flat \end{bmatrix}. \tag{3}$$

The standard formalism of linear algebra now yields the product of dual scalars as

$$\boldsymbol{\gamma} = \boldsymbol{\alpha}\boldsymbol{\beta} = \begin{bmatrix} \alpha^\flat & \alpha^\sharp \\ 0 & \alpha^\flat \end{bmatrix} \begin{bmatrix} \beta^\flat & \beta^\sharp \\ 0 & \beta^\flat \end{bmatrix} = \begin{bmatrix} \alpha^\flat\beta^\flat & \alpha^\flat\beta^\sharp + \alpha^\sharp\beta^\flat \\ 0 & \alpha^\flat\beta^\flat \end{bmatrix}. \tag{4}$$

Here again, the rules associated with bookkeeping parameter ϵ have been "hard wired" in the structure of matrix $\boldsymbol{\alpha}$. The n^{th} power of a dual scalar is obtained easily

$$\boldsymbol{\alpha}^n = \begin{bmatrix} (\alpha^\flat)^n & n\alpha^\sharp(\alpha^\flat)^{n-1} \\ 0 & (\alpha^\flat)^n \end{bmatrix}. \tag{5}$$

The proposed notation circumvents the need for bookkeeping parameter ϵ and its associated rules; the formalism of linear algebra suffices. The inverse operation, which extracts the primal and dual parts from the dual scalar, is noted $\langle\alpha^\flat, \alpha^\sharp\rangle = \text{axial}(\boldsymbol{\alpha})$; for instance, eq. (5) implies $\text{axial}(\boldsymbol{\alpha}^n) = \langle(\alpha^\flat)^n, n\alpha^\sharp(\alpha^\flat)^{n-1}\rangle$.

Similar entities can be formed with quantities that are not scalars. For instance, two vectors of size 3×1, $\langle\underline{a}^\flat, \underline{a}^\sharp\rangle$, and two matrices of size $n \times m$, $\langle\underline{\underline{A}}^\flat, \underline{\underline{A}}^\sharp\rangle$, are used to form two matrices

$$\underline{\boldsymbol{a}} = \begin{bmatrix} \underline{a}^\flat & \underline{a}^\sharp \\ \underline{0} & \underline{a}^\flat \end{bmatrix}, \text{ and } \underline{\underline{\boldsymbol{A}}} = \begin{bmatrix} \underline{\underline{A}}^\flat & \underline{\underline{A}}^\sharp \\ \underline{\underline{0}} & \underline{\underline{A}}^\flat \end{bmatrix}, \tag{6}$$

respectively, where matrices $\underline{\boldsymbol{a}}$ and $\underline{\underline{\boldsymbol{A}}}$, of size 6×2 and $2n \times 2m$, respectively, are referred to as dual vectors and dual matrices, respectively. Notation $(\cdot)^\dagger$ is used to indicate the following matrices

$$\underline{\boldsymbol{a}}^\dagger = \begin{bmatrix} \underline{a}^{\flat T} & \underline{a}^{\sharp T} \\ \underline{0}^T & \underline{a}^{\flat T} \end{bmatrix}, \quad \underline{\underline{\boldsymbol{A}}}^\dagger = \begin{bmatrix} \underline{\underline{A}}^{\flat T} & \underline{\underline{A}}^{\sharp T} \\ \underline{\underline{0}} & \underline{\underline{A}}^{\flat T} \end{bmatrix}, \tag{7}$$

where matrices $\underline{\boldsymbol{a}}^\dagger$ and $\underline{\underline{\boldsymbol{A}}}^\dagger$ are of size 2×6 and $2m \times 2n$, respectively. Notation $(\cdot)^\dagger$ should not be confused with the matrix transposition operation: indeed, its definition implies that $\boldsymbol{\alpha}^\dagger = \boldsymbol{\alpha}$ and $\underline{\boldsymbol{a}}^\dagger \neq \underline{\boldsymbol{a}}^T$. Note that dual scalars commute with dual vectors and matrices, *i.e.*, $\boldsymbol{\alpha}\underline{\boldsymbol{b}} = \underline{\boldsymbol{b}}\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}\underline{\underline{\boldsymbol{A}}} = \underline{\underline{\boldsymbol{A}}}\boldsymbol{\alpha}$.

Similarly, given two vectors, $\underline{u}^\flat$ and $\underline{u}^\sharp$, dual skew-symmetric matrices are formed as

$$\tilde{\boldsymbol{u}} = \begin{bmatrix} \underline{\underline{\tilde{u}}}^\flat & \underline{\underline{\tilde{u}}}^\sharp \\ \underline{\underline{0}} & \underline{\underline{\tilde{u}}}^\flat \end{bmatrix}, \quad \tilde{\boldsymbol{u}}^\dagger = \begin{bmatrix} \underline{\underline{\tilde{u}}}^{\flat T} & \underline{\underline{\tilde{u}}}^{\sharp T} \\ \underline{\underline{0}}^T & \underline{\underline{\tilde{u}}}^{\flat T} \end{bmatrix}. \tag{8}$$

Finally, given two quaternions, $\hat{e}^\flat = \{\mu^\flat, e_1^\flat, e_2^\flat, e_3^\flat\}$ and $\hat{e}^\sharp = \{\mu^\sharp, e_1^\sharp, e_2^\sharp, e_3^\sharp\}$, dual quaternions are defined,

$$\hat{\boldsymbol{e}} = \begin{bmatrix} \hat{e}^\flat & \hat{e}^\sharp \\ \hat{0} & \hat{e}^\flat \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu^\flat & \mu^\sharp \\ 0 & \mu^\flat \end{bmatrix}, \quad \underline{\boldsymbol{e}} = \begin{bmatrix} \underline{e}^\flat & \underline{e}^\sharp \\ \underline{0} & \underline{e}^\flat \end{bmatrix}. \tag{9}$$

The scalar and vector parts of the two quaternions form dual scalars and vectors, respectively, denoted $\boldsymbol{\mu}$ and $\underline{\boldsymbol{e}}$, respectively.

The trace of a dual matrix is a dual scalar, $\boldsymbol{\alpha} = \mathrm{tr}(\underline{\underline{\boldsymbol{A}}})$, such that

$$\mathrm{axial}(\boldsymbol{\alpha}) = \langle \mathrm{tr}(\underline{\underline{A}}^\flat), \mathrm{tr}(\underline{\underline{A}}^\sharp) \rangle. \tag{10}$$

The symmetric and skew-symmetric part of a dual matrix are defined in the usual manner

$$\mathrm{symm}(\underline{\underline{\boldsymbol{A}}}) = (\underline{\underline{\boldsymbol{A}}} + \underline{\underline{\boldsymbol{A}}}^\dagger)/2, \tag{11a}$$

$$\mathrm{skew}(\underline{\underline{\boldsymbol{A}}}) = (\underline{\underline{\boldsymbol{A}}} - \underline{\underline{\boldsymbol{A}}}^\dagger)/2, \tag{11b}$$

$$\mathrm{axial}(\underline{\underline{\boldsymbol{A}}}) = \underline{\boldsymbol{a}} \Longleftrightarrow \tilde{\boldsymbol{a}} = \mathrm{skew}(\underline{\underline{\boldsymbol{A}}}). \tag{11c}$$

The proposed notation is based on the specific structure of the 2×2 matrix defined by eq. (3). This matrix can be viewed as a container in which various entities can be arranged in a specific pattern: dual scalars, vectors, matrices, and quaternions all follow the same structure. The following mnemonic helps the differentiation of the various quantities: bold Greek symbols, lower case letters, and upper case letters refer to dual scalars, vectors, and matrices, respectively. Dual vectors and matrices are underline once and twice, respectively, to indicate the nature of their constituent components. The proposed matrix notation of dual entities eliminates the need for bookkeeping parameter ϵ and associated rules: manipulation of dual entities follows the well-known rules of linear algebra.

2.1 Functions of dual variables

A function of a dual variable is itself a dual scalar written as $\boldsymbol{\theta} = \boldsymbol{\theta}(\boldsymbol{\alpha})$, or more explicitly, $\theta^\flat = \theta^\flat(\alpha^\flat, \alpha^\sharp)$ and $\theta^\sharp = \theta^\sharp(\alpha^\flat, \alpha^\sharp)$. In complex calculus, the real and imaginary parts of the function both depend on the real and imaginary part of the complex variable; clearly, functions of dual and complex variables are similar. The dual functions to be used here are required to be analytic [28], which implies that they can be written as

$$\boldsymbol{\theta}(\boldsymbol{\alpha}) = \sum_{n=0}^{\infty} c_n (\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)^n, \tag{12}$$

for any $\boldsymbol{\alpha}_0$. Using eq. (5) to express the powers of the dual scalar leads to $\theta^\flat = \sum_{n=0}^{\infty} c_n(\alpha^\flat - \alpha_0^\flat)^n$, which implies that $\theta^\flat = \theta^\flat(\alpha^\flat)$ is a real analytic function of variable $\alpha^\flat$ only, and $\theta^\sharp = (\alpha^\sharp - \alpha_0^\sharp)\sum_{n=0}^{\infty} nc_n(\alpha^\flat - \alpha_0^\flat)^{n-1} = \alpha^\sharp\theta^{\flat\prime}$, which implies $\theta^\sharp = \alpha^\sharp\theta^{\flat\prime}$, where notation $(\cdot)'$ indicates a derivative with respect to $\alpha^\flat$.

In summary, analytic dual functions must present the following form

$$\text{axial}(\boldsymbol{\theta}(\boldsymbol{\alpha})) = \langle \theta^\flat, \alpha^\sharp\theta^{\flat\prime}\rangle. \tag{13}$$

If notation $(\cdot)^+$ indicates a derivative with respect to dual scalar $\boldsymbol{\alpha}$, eq. (12) yields $\boldsymbol{\theta}^+ = \sum_{n=0}^{\infty} nc_n(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)^{n-1}$ and expanding the powers of the dual scalar yields

$$\text{axial}(\boldsymbol{\theta}^+(\boldsymbol{\alpha})) = \langle \theta^{\flat\prime}, \alpha^\sharp\theta^{\flat\prime\prime}\rangle. \tag{14}$$

More details about functions of dual variables are found in appendix A.

2.2 Common operations and identities

Let $\underline{p}$ and $\underline{q}$ be two dual vectors; vector operations, such as the scalar, vector, and tensor products of two vectors, are generalized to dual vectors easily,

$$\underline{p}^\dagger\underline{q} = \boldsymbol{\alpha}, \quad \text{axial}(\boldsymbol{\alpha}) = \langle \underline{p}^{\flat T}\underline{q}^\flat, \underline{p}^{\flat T}\underline{q}^\sharp + \underline{p}^{\sharp T}\underline{q}^\flat\rangle, \tag{15a}$$

$$\tilde{\underline{p}}\underline{q} = \underline{u}, \quad \text{axial}(\underline{u}) = \langle \tilde{p}^\flat\underline{q}^\flat, \tilde{p}^\flat\underline{q}^\sharp + \tilde{p}^\sharp\underline{q}^\flat\rangle, \tag{15b}$$

$$\underline{p}\,\underline{q}^\dagger = \underline{\underline{\mathcal{A}}}, \quad \text{axial}(\underline{\underline{\mathcal{A}}}) = \langle \underline{p}^\flat\underline{q}^{\flat T}, \underline{p}^\flat\underline{q}^{\sharp T} + \underline{p}^\sharp\underline{q}^{\flat T}\rangle. \tag{15c}$$

Equation (15a) shows that the scalar product of dual vectors is a dual scalar, *i.e.*, a 2×2 matrix. The square of the norm of a vector (a scalar), $\underline{p}^T\underline{p}$, generalizes to the square of the norm of a dual vector, (a dual scalar), $\boldsymbol{\alpha} = \underline{p}^\dagger\underline{p}$, where $\text{axial}(\boldsymbol{\alpha}) = \langle \|\underline{p}^\flat\|^2, 2\underline{p}^{\flat T}\underline{p}^\sharp\rangle$. A unit vector is such that $\bar{p}^T\bar{p} = 1$; a unit dual vector is such that $\bar{\underline{p}}^\dagger\bar{\underline{p}} = \boldsymbol{\iota}$, where $\boldsymbol{\iota}$ is the unit dual scalar, $\text{axial}(\boldsymbol{\iota}) = \langle 1, 0\rangle$. Hence, a unit dual vector is such that vector $\underline{p}^\flat$ is unit ($\|\bar{p}^\flat\|^2 = 1$) and vectors $\bar{p}^\flat$ and $\underline{p}^\sharp$ are orthogonal ($\bar{p}^{\flat T}\underline{p}^\sharp = 0$).

The following dual vector identities are verified easily,

$$\tilde{\underline{p}}\underline{q} = -\tilde{\underline{q}}\underline{p}, \tag{16a}$$

$$\tilde{\underline{p}}\tilde{\underline{q}} = \underline{q}\,\underline{p}^\dagger - (\underline{p}^\dagger\underline{q})\underline{\underline{I}}, \tag{16b}$$

$$\widetilde{\tilde{\underline{p}}\underline{q}} = \tilde{\underline{p}}\tilde{\underline{q}} - \tilde{\underline{q}}\tilde{\underline{p}}, \tag{16c}$$

$$\text{tr}(\tilde{\underline{p}}) = 0, \tag{16d}$$

$$\text{tr}(\tilde{\underline{p}}\tilde{\underline{q}}) = -2\underline{p}^\dagger\underline{q}, \tag{16e}$$

where $\underline{\underline{I}}$ is the identity dual matrix, $\text{axial}(\underline{\underline{I}}) = \langle \underline{\underline{I}}, \underline{\underline{0}}\rangle$. These identities generalize the common vector identities $\tilde{p}\underline{q} = -\tilde{q}\underline{p}$, $\tilde{p}\tilde{q} = \underline{q}\,\underline{p}^T - (\underline{p}^T\underline{q})\underline{\underline{I}}$, $\widetilde{\tilde{p}\underline{q}} = \tilde{p}\tilde{q} - \tilde{q}\tilde{p}$, etc. If $\bar{\underline{n}}$ is a unit dual vector, the following identities also hold

$$\tilde{\underline{n}}\tilde{\underline{n}}\tilde{\underline{n}} = -\tilde{\underline{n}}. \tag{17a}$$

$$\tilde{\underline{n}}\dot{\tilde{\underline{n}}}\tilde{\underline{n}} = \underline{\underline{0}}, \tag{17b}$$

where notation $(\dot{\cdot})$ indicates a derivative with respect to time.

Matrix $\underline{\underline{A}}$ is orthogonal if $\underline{\underline{A}}^T\underline{\underline{A}} = \underline{\underline{I}}$. By analogy, a dual matrix is said to be orthogonal if $\underline{\underline{\mathcal{A}}}^\dagger\underline{\underline{\mathcal{A}}} = \underline{\underline{I}}$. This condition implies $\underline{\underline{A}}^{\flat T}\underline{\underline{A}}^\flat = \underline{\underline{I}}$ and $(\underline{\underline{A}}^{\flat T}\underline{\underline{A}}^\sharp) + (\underline{\underline{A}}^{\flat T}\underline{\underline{A}}^\sharp)^T = \underline{\underline{0}}$, or $(\underline{\underline{A}}^{\flat T}\underline{\underline{A}}^\sharp)$ is skew-symmetric.

As an application, consider the following statement $\tilde{\underline{a}}\underline{x} = \underline{b}$ that can be viewed as a linear system for unknown $\underline{x}$. Because $\tilde{\underline{a}}\underline{a} = \underline{0}$, the system is singular twice and $\underline{a}$ forms the null space of the system; the solvability condition is $\underline{a}^\dagger \underline{b} = \mathbf{o}$. The solution of the problem is $\underline{x} = \boldsymbol{\lambda}\underline{a} + \boldsymbol{\alpha}\tilde{\underline{a}}\underline{b}$, where $\boldsymbol{\lambda}$ is an arbitrary dual scalar and $\boldsymbol{\alpha}$ a dual scalar to be solved for. Introducing the solution yields $\underline{b} = \tilde{\underline{a}}(\boldsymbol{\lambda}\underline{a} + \boldsymbol{\alpha}\tilde{\underline{a}}\underline{b}) = \boldsymbol{\alpha}\tilde{\underline{a}}\tilde{\underline{a}}\underline{b}$. Identity (16b) yields $\underline{b} = \boldsymbol{\alpha}[\underline{a}\,\underline{a}^\dagger - (\underline{a}^\dagger \underline{a})\underline{\underline{I}}]\underline{b} = -\boldsymbol{\alpha}(\underline{a}^\dagger \underline{a})\underline{b}$, where the solvability condition implies the second equality. Clearly, $\boldsymbol{\alpha} = -\boldsymbol{\iota}/(\underline{a}^\dagger \underline{a})$ and the solution is

$$\underline{x} = \boldsymbol{\lambda}\underline{a} - \frac{\tilde{\underline{a}}\underline{b}}{(\underline{a}^\dagger \underline{a})}, \tag{18}$$

where dual scalar $\boldsymbol{\lambda}$ remains undetermined, as expected.

2.3 Plücker coordinates

The Plücker coordinates of a line consist of a unit vector, $\bar{p}$, defining its orientation and vector $\tilde{x}\bar{p}$, where $\underline{x}$ is the position vector of any point on the line. Dual vectors provide a convenient representation of Plücker coordinates: $\bar{\underline{p}}$, such that $\text{axial}(\bar{\underline{p}}) = \langle \bar{p}, \tilde{x}\bar{p}\rangle$. Note that vector $\bar{p}$ and dual vector $\bar{\underline{p}}$ are both unit.

The scalar product of two unit vectors defines the cosine of the angle, α, between the two vectors, $\bar{p}^T\bar{q} = \cos\alpha$. Similarly, the scalar product of dual vectors is $\bar{\underline{p}}^\dagger\bar{\underline{q}} = \cos\boldsymbol{\alpha}$: indeed, $\text{axial}(\bar{\underline{p}}^\dagger\bar{\underline{q}}) = \langle \bar{p}^T\bar{q}, \bar{p}^T\tilde{x}_Q\bar{q} + \bar{q}^T\tilde{x}\bar{p}\rangle = \langle \cos\alpha, -\delta\sin\alpha\rangle$, where δ is the distance between the two lines. Defining dual scalar $\boldsymbol{\alpha}$ such that $\text{axial}(\boldsymbol{\alpha}) = \langle \alpha, \delta\rangle$, $\cos\boldsymbol{\alpha}$ is an analytic dual function, see eq. (13). The condition $\bar{p}^T\bar{q} = 0$, which implies the normality of vectors $\bar{p}$ and $\bar{q}$, generalizes to $\bar{\underline{p}}^\dagger\bar{\underline{q}} = \mathbf{o}$, which implies that lines $\bar{\underline{p}}$ and $\bar{\underline{q}}$ are mutually orthogonal, intersecting lines.

The vector product of two unit vectors defines the sine of the angle between the two vectors, $\tilde{\bar{p}}\bar{q} = \sin\alpha\,\bar{n}$, where $\bar{n}$ is the unit vector normal to vectors $\bar{p}$ and $\bar{q}$ and oriented according to the right hand rule. Similarly, the vector product $\tilde{\bar{\underline{p}}}\bar{\underline{q}} = \sin\boldsymbol{\alpha}\,\bar{\underline{n}}$, where $\sin\boldsymbol{\alpha}$ is an analytic dual function. Because $\bar{\underline{n}}^\dagger\bar{\underline{p}} = \bar{\underline{n}}^\dagger\bar{\underline{q}} = \underline{0}$, line $\bar{\underline{n}}$ is normal unit vectors $\bar{p}$ and $\bar{q}$ and intersects lines $\bar{\underline{p}}$ and $\bar{\underline{q}}$. This means that line $\bar{\underline{n}}$ joins the point of lines $\bar{\underline{p}}$ and $\bar{\underline{q}}$ that are at the shortest distance from each other.

A basis is formed by a set of three mutually orthogonal unit vectors, $\mathcal{B} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$. This generalizes to a frame, which consists of three mutually orthogonal and intersection lines, $\mathcal{F} = (\bar{\underline{\imath}}_1, \bar{\underline{\imath}}_2, \bar{\underline{\imath}}_3)$. Common relationships such as $\tilde{\bar{\imath}}_1\bar{\imath}_2 = \bar{\imath}_3$ generalize to $\tilde{\bar{\underline{\imath}}}_1\bar{\underline{\imath}}_2 = \bar{\underline{\imath}}_3$. Because identity $\bar{\imath}_1\bar{\imath}_1^T + \bar{\imath}_2\bar{\imath}_2^T + \bar{\imath}_3\bar{\imath}_3^T = \underline{\underline{I}}$ has its counterpart as $\bar{\underline{\imath}}_1\bar{\underline{\imath}}_1^\dagger + \bar{\underline{\imath}}_2\bar{\underline{\imath}}_2^\dagger + \bar{\underline{\imath}}_3\bar{\underline{\imath}}_3^\dagger = \underline{\underline{I}}$, the components of vector $\underline{a}$ resolved in basis $\mathcal{B}$ and those of line $\underline{a}$ resolved in frame $\mathcal{F}$ are $\bar{\imath}_k^T\underline{a}$ and $\bar{\underline{\imath}}_k^\dagger\underline{a}$, respectively.

3 Geometric description of motion

Chasles' theorem [29] states that the most general motion of a rigid body consists of a translation along a line followed by a rotation about the same line. Hence, a general motion is characterized by its Chasles' line of Plücker coordinates $\bar{\underline{n}}$, and the magnitudes of the rotation and intrinsic displacement, denoted ϕ and δ, respectively, for a total of six parameters. The scalar characteristics of the motion form a dual scalar, $\boldsymbol{\phi} = \langle \phi, \delta\rangle$. In this section, the basic formulæ required for the manipulation of motion are expressed in terms of geometric entities $(\bar{\underline{n}}, \boldsymbol{\phi})$.

3.1 The motion tensor

The Plücker coordinates of a material line of the body before and after it undergoes the specified motion are denoted $\bar{\underline{a}}$ and $\bar{\underline{b}}$, respectively, as shown in fig. 1. The following question is raised: what is the relationship between Plücker coordinates $\bar{\underline{a}}$ and $\bar{\underline{b}}$?

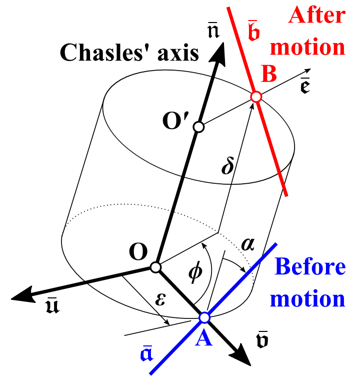

Figure 1: Material line of a body before and after motion

Vector product $\tilde{\boldsymbol{n}}\bar{\boldsymbol{a}} = \sin\boldsymbol{\alpha}\bar{\boldsymbol{v}}$ defines line $\bar{\boldsymbol{v}}$ that is perpendicular to and intersects lines $\bar{\boldsymbol{n}}$ and $\bar{\boldsymbol{a}}$ at points $\mathbf{O}$ and $\mathbf{A}$, respectively. Dual scalar $\boldsymbol{\alpha} = \langle \alpha, \varepsilon \rangle$ defines the angle α between the lines and their shortest distance, ε, see fig 1. Next, vector product $\tilde{\boldsymbol{v}}\bar{\boldsymbol{n}} = \bar{\boldsymbol{u}}$ defines the last line of the canonical frame of the motion, $\mathcal{F} = (\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{n}})$. Figure 1 shows a cylinder of radius ε and axis $\bar{\boldsymbol{n}}$ coincident with Chasles' line. Line $\bar{\boldsymbol{a}}$ is in the plane tangent to this cylinder at point $\mathbf{A}$. During motion, line $\bar{\boldsymbol{a}}$ rotates around the cylinder by an angle ϕ and translates along line $\bar{\boldsymbol{n}}$ by a distance δ. At the end of the motion, material point $\mathbf{A}$ has moved to point $\mathbf{B}$ and line $\bar{\boldsymbol{b}}$ is in the plane tangent to the cylinder at point $\mathbf{B}$.

Because lines $\bar{\boldsymbol{n}}$ and $\bar{\boldsymbol{a}}$ are material lines of the body, their distance and relative orientation remain unchanged, *i.e.*, $\bar{\boldsymbol{n}}^\dagger\bar{\boldsymbol{a}} = \bar{\boldsymbol{n}}^\dagger\bar{\boldsymbol{b}} = \cos\boldsymbol{\alpha}$. For the same reasons, $\tilde{\boldsymbol{n}}\bar{\boldsymbol{b}} = \sin\boldsymbol{\alpha}\bar{\boldsymbol{e}}$, where line $\bar{\boldsymbol{e}}$ is in the plane normal to $\bar{\boldsymbol{n}}$ at a distance δ from point $\mathbf{O}$, *i.e.*, $\bar{\boldsymbol{e}} = \cos\boldsymbol{\phi}\bar{\boldsymbol{v}} - \sin\boldsymbol{\phi}\bar{\boldsymbol{u}}$. It follows that $\tilde{\boldsymbol{n}}\bar{\boldsymbol{b}} = \sin\boldsymbol{\alpha}(\cos\boldsymbol{\phi}\bar{\boldsymbol{v}} - \sin\boldsymbol{\phi}\bar{\boldsymbol{u}})$, a vector-product equation whose solution is given by eq. (18) as

$$\bar{\boldsymbol{b}} = \boldsymbol{\lambda}\bar{\boldsymbol{n}} - \sin\boldsymbol{\alpha}\ \tilde{\boldsymbol{n}}(\cos\boldsymbol{\phi}\bar{\boldsymbol{v}} - \sin\boldsymbol{\phi}\bar{\boldsymbol{u}}). \tag{19}$$

Dual scalar $\boldsymbol{\lambda}$ is found to be $\boldsymbol{\lambda} = \bar{\boldsymbol{n}}^\dagger\bar{\boldsymbol{b}} = \bar{\boldsymbol{n}}^\dagger\bar{\boldsymbol{a}}$, leading to $\bar{\boldsymbol{b}} = \bar{\boldsymbol{n}}\bar{\boldsymbol{n}}^\dagger\bar{\boldsymbol{a}} + \sin\boldsymbol{\phi}\tilde{\boldsymbol{n}}\bar{\boldsymbol{a}} - \cos\boldsymbol{\phi}\tilde{\boldsymbol{n}}\tilde{\boldsymbol{n}}\bar{\boldsymbol{a}}$. Finally, identity (16b) yields the desired result, $\bar{\boldsymbol{b}} = \underline{\underline{\mathcal{R}}}\bar{\boldsymbol{a}}$, where

$$\underline{\underline{\mathcal{R}}}(\bar{\boldsymbol{n}}, \boldsymbol{\phi}) = \underline{\underline{\mathcal{I}}} + \sin\boldsymbol{\phi}\tilde{\boldsymbol{n}} + (\boldsymbol{\iota} - \cos\boldsymbol{\phi})\tilde{\boldsymbol{n}}\tilde{\boldsymbol{n}}, \tag{20}$$

is the motion tensor, $\text{axial}(\underline{\underline{\mathcal{R}}}) = \langle \underline{\underline{R}}^\flat, \underline{\underline{R}}^\sharp \rangle$, which is fully defined by geometric entities $(\bar{\boldsymbol{n}}, \boldsymbol{\phi})$. The motion tensor can also be expressed as a matrix exponential

$$\underline{\underline{\mathcal{R}}} = \exp(\boldsymbol{\phi}\tilde{\boldsymbol{n}}) = \sum_{k=0}^{\infty} \frac{\boldsymbol{\phi}^n\tilde{\boldsymbol{n}}^k}{k!}, \tag{21}$$

called the exponential map of the motion tensor. Starting from eq. (20), the sine and cosine functions are expanded in Taylor series using eq. (71) and identity (17a) then yields the exponential map.

The notation adopted in this paper shows that matrix $\underline{\underline{R}}^\flat$ relates the orientations of the lines $\bar{\boldsymbol{a}}$ and $\bar{\boldsymbol{b}}$ as $\bar{b} = \underline{\underline{R}}^\flat\bar{a}$, where

$$\underline{\underline{R}}^\flat = \exp(\phi\tilde{n}) = \underline{\underline{I}} + \sin\phi\tilde{n} + (1 - \cos\phi)\tilde{n}\tilde{n}. \tag{22}$$

This expression is, of course, Rodrigues' formula for the rotation tensor and eq. (20) generalizes this formula to the Plücker coordinates of the corresponding lines. Rotation becomes a subset of motion, as should be.

With the help of identity (17a), it can be shown that the motion tensor is an orthogonal dual matrix, $\underline{\underline{\mathcal{R}}}^\dagger\underline{\underline{\mathcal{R}}} = \underline{\underline{\mathcal{I}}}$, which implies the orthogonality of the rotation tensor, $\underline{\underline{R}}^{\flat T}\underline{\underline{R}}^\flat = \underline{\underline{I}}$ and the fact that matrix $\underline{\underline{R}}^{\flat T}\underline{\underline{R}}^{\sharp}$ is skew-symmetric. Tedious algebra reveals that $\underline{\underline{R}}^{\flat T}\underline{\underline{R}}^{\sharp} = \tilde{z}$, where $\underline{z} = \delta\bar{n} + (\underline{\underline{R}}^{\flat T} - \underline{\underline{I}})\underline{x}$ and $\underline{x}$ is the position vector of an arbitrary point on Chasles' line. It follows that $\underline{\underline{R}}^{\sharp} = \underline{\underline{R}}^{\flat}\tilde{z} = \tilde{u}\underline{\underline{R}}^{\flat}$, where $\underline{u} = \underline{\underline{R}}^{\flat}\underline{z}$, leading to

$$\underline{u} = \delta\bar{n} + (\underline{\underline{I}} - \underline{\underline{R}}^{\flat})\underline{x} = \delta\bar{n} + 2\sin\frac{\phi}{2}(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\tilde{n})\underline{m}, \tag{23}$$

which is the displacement vector of the point of rigid body that coincides with the origin of the reference frame. In summary, the motion tensor is an orthogonal dual matrix that can be expressed as axial$(\underline{\underline{\mathcal{R}}}) = \langle \underline{\underline{R}}^{\flat}, \tilde{u}\underline{\underline{R}}^{\flat} \rangle$.

It will be convenient to define dual vector

$$\underline{e} = \sin\frac{\boldsymbol{\phi}}{2}\bar{n}, \tag{24}$$

with axial$(\underline{e}) = \langle \underline{e}^{\flat}, \underline{e}^{\sharp} \rangle$. Using elementary trigonometric formulæ, the motion tensor can then be recast as

$$\underline{\underline{\mathcal{R}}} = \underline{\underline{I}} + 2\cos\frac{\boldsymbol{\phi}}{2}\tilde{e} + 2\tilde{e}\tilde{e}, \tag{25}$$

and the displacement vector of the point of rigid body that coincides with the origin of the reference frame becomes

$$\underline{u} = \delta\sin\frac{\phi}{2}\underline{e}^{\flat} + 2\cos\frac{\phi}{2}\underline{e}^{\sharp} + 2\tilde{e}^{\flat}\underline{e}^{\sharp}. \tag{26}$$

The following properties of the motion tensor are verified easily

$$\mathrm{tr}(\underline{\underline{\mathcal{R}}}) = \boldsymbol{\iota} + 2\cos\boldsymbol{\phi}, \tag{27a}$$

$$\mathrm{axial}(\underline{\underline{\mathcal{R}}}) = \bar{n}\sin\boldsymbol{\phi} = 2\bar{e}\cos\boldsymbol{\phi}/2, \tag{27b}$$

$$\mathrm{symm}(\underline{\underline{\mathcal{R}}}) = \cos\boldsymbol{\phi} + (\boldsymbol{\iota} - \cos\boldsymbol{\phi})\bar{n}\bar{n}^{\dagger} = (\mathrm{tr}(\underline{\underline{\mathcal{R}}}) - \boldsymbol{\iota})/2 + 2\bar{e}\bar{e}^{\dagger}. \tag{27c}$$

Two multiplicative decompositions of the motion tensor are now presented. Consider the following fractional motion tensor

$$\underline{\underline{\mathcal{G}}}_m(\bar{n}, \frac{\boldsymbol{\phi}}{m}) = \underline{\underline{I}} + \sin\frac{\boldsymbol{\phi}}{m}\tilde{n} + (\boldsymbol{\iota} - \cos\frac{\boldsymbol{\phi}}{m})\tilde{n}\tilde{n}, \tag{28}$$

where m is a positive integer. Elementary trigonometric identities and identity (17a) then yield $\underline{\underline{\mathcal{R}}} = \underline{\underline{\mathcal{G}}}_m^m$. Clearly, fractional motion tensor $\underline{\underline{\mathcal{G}}}_m$ decomposes the motion into a sequence of m fractional motions, each of magnitude $\boldsymbol{\phi}/m$, all about the same Chasles' line $\bar{n}$. Motion tensor $\underline{\underline{\mathcal{G}}}_m$ is the m^{th} root of motion tensor $\underline{\underline{\mathcal{R}}}$.

The second multiplicative decomposition of the motion tensor, know as Cayley's decomposition, is

$$\underline{\underline{\mathcal{R}}} = (\underline{\underline{I}} + \tan\frac{\boldsymbol{\phi}}{2}\tilde{n})(\underline{\underline{I}} - \tan\frac{\boldsymbol{\phi}}{2}\tilde{n})^{-1} = (\underline{\underline{I}} - \tan\frac{\boldsymbol{\phi}}{2}\tilde{n})^{-1}(\underline{\underline{I}} + \tan\frac{\boldsymbol{\phi}}{2}\tilde{n}), \tag{29}$$

which follows from elementary trigonometric identities and identity (17a). Applying Cayley's decomposition to motion tensor $\underline{\underline{\mathcal{G}}}_m$ yields $\underline{\underline{\mathcal{G}}}_m = (\underline{\underline{I}} + \tan\boldsymbol{\phi}/(2m)\tilde{n})(\underline{\underline{I}} - \tan\boldsymbol{\phi}/(2m)\tilde{n})^{-1}$, leading to Cayley's higher-order decomposition [30, 31]

$$\underline{\underline{\mathcal{R}}} = (\underline{\underline{I}} + \tan\frac{\boldsymbol{\phi}}{2m}\tilde{n})^{m}(\underline{\underline{I}} - \tan\frac{\boldsymbol{\phi}}{2m}\tilde{n})^{-m} = (\underline{\underline{I}} - \tan\frac{\boldsymbol{\phi}}{2m}\tilde{n})^{-m}(\underline{\underline{I}} + \tan\frac{\boldsymbol{\phi}}{2m}\tilde{n})^{m}. \tag{30}$$

Clearly, eq. (29) is the first-order ($m = 1$) decomposition, a special case of the more general decomposition (30).

3.2 The canonical frame

In eq. (20) the Plücker coordinates of Chasles' line can be evaluated in any frame. On the other hand, fig. 1 shows that the frame defined by mutually intersection normal lines $\bar{u}$, $\bar{v}$, and $\bar{n}$, called the *canonical frame*, is well suited to the description of the motion. Line $\bar{n}$ is Chasles' axis and lines $\bar{u}$ and $\bar{v}$ can be chosen arbitrarily in the plane normal to Chasles' axis. When resolved in the canonical frame, the Plücker coordinates of Chasles' line are $\text{axial}(\bar{n}) = \langle \bar{\imath}_1, \underline{0} \rangle$, leading to

$$\underline{\underline{R}}^\flat = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{\underline{R}}^\sharp = \delta \begin{bmatrix} -S_\phi & -C_\phi & 0 \\ C_\phi & -S_\phi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{31}$$

where $S_\phi = \sin\phi$ and $C_\phi = \cos\phi$. As expected, rotation matrix $\underline{\underline{R}}^\flat$ expresses a planar rotation of magnitude ϕ about unit vector $\bar{\imath}_3$ and $\underline{\underline{R}}^\sharp = \delta \tilde{\imath}_3 \underline{\underline{R}}^\flat$. When resolved in the canonical frame, the components of the motion tensor only depend on the scalar parameters of the motion, ϕ and δ. Chasles' line defines the axis about which the motion takes place and is built into the definition of the canonical frame. The canonical frame is not defined uniquely: its origin is at any point along Chasles' axis and lines $\bar{u}$ and $\bar{v}$ can rotate freely about the same axis.

3.3 The velocity vector

Let $\underline{\underline{\mathcal{R}}}(t)$ be the time-dependent motion tensor that brings inertial frame $\mathcal{F}_I = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ to frame $\mathcal{F}(t) = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$, which implies $\bar{e}_1(t) = \underline{\underline{\mathcal{R}}}(t)\bar{\imath}_1$. Because the motion tensor is an orthogonal dual matrix, a time derivative of the orthogonality statement yields $(\dot{\underline{\underline{\mathcal{R}}}}\,\underline{\underline{\mathcal{R}}}^\dagger) + (\dot{\underline{\underline{\mathcal{R}}}}\,\underline{\underline{\mathcal{R}}}^\dagger)^\dagger = \underline{\underline{0}}$, which shows that the dual matrix in the parentheses must be a skew-symmetric dual matrix

$$\tilde{v} = \dot{\underline{\underline{\mathcal{R}}}}\,\underline{\underline{\mathcal{R}}}^\dagger. \tag{32}$$

Dual vector $\underline{v}$ is the velocity vector, which enables the evaluation of the time derivative of line $\bar{e}_1$ as $\dot{\bar{e}}_1 = \tilde{v}\bar{e}_1$. It follows that the time derivative of the Plücker coordinates, $\bar{a}$, of any material line of the body is $\dot{\bar{a}} = \tilde{v}\bar{a}$.

The velocity vector can be expressed in terms of geometric entities $(\bar{n}, \boldsymbol{\phi})$ and their time derivatives. Introducing eq. (20) into eq. (32) and using identities (17a) and (17b) leads to

$$\underline{v} = \dot{\boldsymbol{\phi}}\;\bar{n} + \sin\boldsymbol{\phi}\;\dot{\bar{n}} + (\boldsymbol{\iota} - \cos\boldsymbol{\phi})\tilde{\bar{n}}\dot{\bar{n}}, \tag{33a}$$

$$\underline{v}^* = \dot{\boldsymbol{\phi}}\;\bar{n} + \sin\boldsymbol{\phi}\;\dot{\bar{n}} - (\boldsymbol{\iota} - \cos\boldsymbol{\phi})\tilde{\bar{n}}\dot{\bar{n}}, \tag{33b}$$

where $\underline{v}^* = \underline{\underline{\mathcal{R}}}^\dagger \underline{v}$ are the components of the dual velocity vector resolved in the material frame. Using dual vector $\underline{e}$ defined by eq. (24), the velocity vector becomes

$$\underline{v} = \dot{\boldsymbol{\phi}} \sin\frac{\boldsymbol{\phi}}{2}\underline{e} + 2\cos\frac{\boldsymbol{\phi}}{2}\dot{\underline{e}} + 2\tilde{\underline{e}}\dot{\underline{e}}, \tag{34a}$$

$$\underline{v}^* = \dot{\boldsymbol{\phi}} \sin\frac{\boldsymbol{\phi}}{2}\underline{e} + 2\cos\frac{\boldsymbol{\phi}}{2}\dot{\underline{e}} - 2\tilde{\underline{e}}\dot{\underline{e}}. \tag{34b}$$

An alternative expression of the velocity vector is found from the more compact definition of the motion tensor, $\text{axial}(\underline{\underline{\mathcal{R}}}) = \langle \underline{\underline{R}}^\flat, \tilde{u}\underline{\underline{R}}^\flat \rangle$, leading to $\text{axial}(\underline{v}) = \langle \underline{v}^\flat, \underline{v}^\sharp \rangle$. The primal part of the dual velocity vector is the angular velocity vector, $\underline{v}^\flat = \text{axial}(\dot{\underline{\underline{R}}}^\flat \underline{\underline{R}}^{\flat T})$, while its dual part, $\underline{v}^\sharp = \dot{\underline{u}} + \tilde{u}\underline{v}^\flat$, is the velocity of the point of the rigid body that instantaneously coincides with the origin of frame $\mathcal{F}_I$.

3.4 Composition of motion

Consider three orthogonal dual matrices $\underline{\underline{\mathcal{R}}}(\bar{n}, \boldsymbol{\phi})$, $\underline{\underline{\mathcal{R}}}(\bar{n}_1, \boldsymbol{\phi}_1)$, and $\underline{\underline{\mathcal{R}}}(\bar{n}_2, \boldsymbol{\phi}_2)$ such that $\underline{\underline{\mathcal{R}}}(\bar{n}, \boldsymbol{\phi}) = \underline{\underline{\mathcal{R}}}(\bar{n}_1, \boldsymbol{\phi}_1)\underline{\underline{\mathcal{R}}}(\bar{n}_2, \boldsymbol{\phi}_2)$. What is the relationship between the parameters of the composed motion $(\bar{n}, \boldsymbol{\phi})$ and those of the two motions, $(\bar{n}_1, \boldsymbol{\phi}_1)$ and $(\bar{n}_2, \boldsymbol{\phi}_2)$? Letting $\underline{e} = \bar{n} \sin \boldsymbol{\phi}/2$, $\underline{e}_1 = \bar{n}_1 \sin \boldsymbol{\phi}_1/2$, and $\underline{e}_2 = \bar{n}_2 \sin \boldsymbol{\phi}_2/2$, the use eq. (25) leads to

$$\cos \frac{\boldsymbol{\phi}}{2} = \cos \frac{\boldsymbol{\phi}_1}{2} \cos \frac{\boldsymbol{\phi}_2}{2} - \underline{e}_1^T \underline{e}_2, \tag{35a}$$

$$\underline{e} = \cos \frac{\boldsymbol{\phi}_2}{2} \underline{e}_1 + \cos \frac{\boldsymbol{\phi}_1}{2} \underline{e}_2 + \tilde{e}_1 \underline{e}_2. \tag{35b}$$

The first equation yields the scalar parameters of the composed motion, $\boldsymbol{\phi}$, and the second gives Chasles' line, $\underline{e} = \bar{n} \sin \boldsymbol{\phi}/2$.

3.5 Discussion

This section has focused on the geometric description of motion in terms of entities $\bar{n}$ and $\boldsymbol{\phi}$. The motion tensor, velocity vector, and the composition of motion formula were presented in sections 3.1, 3.3, and 3.4, respectively. The motion tensor was derived from basic geometric arguments and all subsequent developments use the identities presented in section 2.2. The formulæ for rotation are a subset of those derived for motion. The notation used in the paper underlines the close relationship between motion and rotation.

4 Euler parameters

Consider quaternion $\hat{e}$ and associated matrices, $\underline{\underline{A}}(\hat{e})$ and $\underline{\underline{B}}(\hat{e})$, defined by eq. (2). A unit quaternion is such that $\|\hat{e}\| = 1$ and for such quaternions, these matrices are orthogonal: $\underline{\underline{A}}^T(\hat{e})\underline{\underline{A}}(\hat{e}) = \underline{\underline{B}}^T(\hat{e})\underline{\underline{B}}(\hat{e}) = \underline{\underline{I}}$. Next consider a dual quaternion $\hat{\boldsymbol{e}}$, as defined by eq. (9). A unit dual quaternion is such that $\hat{\boldsymbol{e}}^\dagger \hat{\boldsymbol{e}} = \boldsymbol{\iota}$. Finally, dual matrix $\underline{\underline{\mathcal{A}}}(\hat{\boldsymbol{e}})$ is introduced such that $\text{axial}(\underline{\underline{\mathcal{A}}}(\hat{\boldsymbol{e}})) = \langle \underline{\underline{A}}(\hat{e}^\flat), \underline{\underline{A}}(\hat{e}^\sharp) \rangle$, with a similar notation for $\underline{\underline{\mathcal{B}}}(\hat{\boldsymbol{e}})$. For unit dual quaternions, these dual matrices are orthogonal: $\underline{\underline{\mathcal{A}}}^\dagger(\hat{\boldsymbol{e}})\underline{\underline{\mathcal{A}}}(\hat{\boldsymbol{e}}) = \underline{\underline{\mathcal{B}}}^\dagger(\hat{\boldsymbol{e}})\underline{\underline{\mathcal{B}}}(\hat{\boldsymbol{e}}) = \underline{\underline{\mathcal{I}}}$.

4.1 The motion tensor

Dual vector $\underline{e}$ defined by eq. (24) is used to construct two dual quaternions,

$$\hat{e}^\flat = \begin{Bmatrix} \mu^\flat \\ \underline{e}^\flat \end{Bmatrix}, \quad \hat{e}^\sharp = \begin{Bmatrix} \mu^\sharp \\ \underline{e}^\sharp \end{Bmatrix}. \tag{36}$$

The scalar parts of these two quaternions are selected so as to render dual quaternion $\hat{\boldsymbol{e}}$ unit. This implies $\|\hat{e}^\flat\| = 1$, and hence, $\mu^\flat = \cos \phi/2$ and $\hat{e}^{\flat T} \hat{e}^\sharp = 0$, leading to $\mu^\sharp = -\delta/2 \sin \phi/2$ and finally,

$$\boldsymbol{\mu} = \cos \frac{\boldsymbol{\phi}}{2}, \tag{37}$$

an analytic dual function.

Starting from eq. (25), the following result is verified easily

$$\underline{\underline{\mathcal{D}}}(\hat{\boldsymbol{e}}) = \underline{\underline{\mathcal{A}}}(\hat{\boldsymbol{e}})\underline{\underline{\mathcal{B}}}^\dagger(\hat{\boldsymbol{e}}) = \underline{\underline{\mathcal{B}}}^\dagger(\hat{\boldsymbol{e}})\underline{\underline{\mathcal{A}}}(\hat{\boldsymbol{e}}), \tag{38}$$

where $\text{axial}(\underline{\underline{\mathcal{D}}}(\hat{e})) = \langle \underline{\underline{D}}^\flat, \underline{\underline{D}}^\sharp \rangle$ and

$$\underline{\underline{D}}^\flat = \begin{bmatrix} 1 & 0^T \\ \underline{0} & \underline{\underline{R}}^\flat \end{bmatrix}, \quad \underline{\underline{D}}^\sharp = \begin{bmatrix} 0 & 0^T \\ \underline{0} & \underline{\underline{R}}^\sharp \end{bmatrix}. \tag{39}$$

Clearly, the motion tensor defined by eq. (25) is a a subset of dual matrix $\underline{\underline{\mathcal{D}}}(\hat{e})$. The two additional lines and columns appearing in matrix $\underline{\underline{\mathcal{D}}}(\hat{e})$ stem from the increased size of dual quaternion $\hat{e}$ compared to dual vector $\underline{e}$ and their entries reflect the fact that dual quaternion $\hat{e}$ is unit.

4.2 The velocity vector

It is also possible to express the velocity vector in term of dual quaternions and their time derivatives. Noting that $\underline{\dot{e}} = \dot{\bar{n}} \sin \boldsymbol{\phi}/2 + \bar{n}\dot{\boldsymbol{\phi}}/2 \cos \boldsymbol{\phi}/2$, eq. (34) becomes

$$\hat{v} = 2\underline{\underline{\mathcal{B}}}^\dagger(\hat{e})\dot{\hat{e}}, \quad \hat{v}^* = 2\underline{\underline{\mathcal{A}}}^\dagger(\hat{e})\dot{\hat{e}}. \tag{40}$$

The velocity dual quaternion, $\hat{v}$, is composed of two quaternions, $\text{axial}(\hat{v}) = \langle \hat{v}^\flat, \hat{v}^\sharp \rangle$, where quaternions $\hat{v}^{\flat T} = \{0, \underline{v}^{\flat T}\}$ and $\hat{v}^{\sharp T} = \{0, \underline{v}^{\sharp T}\}$ vanishing scalar parts.

4.3 Composition of motion

Consider three unit dual quaternions $\hat{e}$, $\hat{e}_1$, and $\hat{e}_2$ such that $\underline{\underline{\mathcal{D}}}(\hat{e}) = \underline{\underline{\mathcal{D}}}(\hat{e}_1)\underline{\underline{\mathcal{D}}}(\hat{e}_2)$. What is the relationship between the composed quaternion $\hat{e}$ and quaternions $\hat{e}_1$ and $\hat{e}_2$? Identity (38) leads to $\underline{\underline{\mathcal{D}}}(\hat{e}) = \underline{\underline{\mathcal{A}}}(\hat{e})\underline{\underline{\mathcal{B}}}^\dagger(\hat{e}) = \underline{\underline{\mathcal{A}}}(\hat{e}_1)\underline{\underline{\mathcal{B}}}^\dagger(\hat{e}_1)\underline{\underline{\mathcal{A}}}(\hat{e}_2)\underline{\underline{\mathcal{B}}}^\dagger(\hat{e}_2) = \underline{\underline{\mathcal{A}}}(\hat{e}_1)\underline{\underline{\mathcal{A}}}(\hat{e}_2)\underline{\underline{\mathcal{B}}}^\dagger(\hat{e}_1)\underline{\underline{\mathcal{B}}}^\dagger(\hat{e}_2)$, which implies $\underline{\underline{\mathcal{A}}}(\hat{e}) = \underline{\underline{\mathcal{A}}}(\hat{e}_1)\underline{\underline{\mathcal{A}}}(\hat{e}_2)$, or equivalently $\underline{\underline{\mathcal{B}}}(\hat{e}) = \underline{\underline{\mathcal{B}}}(\hat{e}_2)\underline{\underline{\mathcal{B}}}(\hat{e}_1)$. Further simplifications yield

$$\hat{e} = \underline{\underline{\mathcal{A}}}(\hat{e}_1)\hat{e}_2 = \underline{\underline{\mathcal{B}}}(\hat{e}_2)\hat{e}_1. \tag{41}$$

4.4 Extraction motion parameters

Equation (38) expresses the motion tensor in terms of Euler parameters. In many applications, the inverse operation is also required, *i.e.*, given the motion tensor, find the Euler parameters. Unfortunately, this inverse operation cannot be written in a simple manner because any such expression will involve a division by a term that can vanish for specific motion tensors.

Consider dual matrix $\underline{\underline{\mathcal{T}}}$, $\text{axial}(\underline{\underline{\mathcal{T}}}) = \langle \underline{\underline{T}}^\flat, \underline{\underline{T}}^\sharp \rangle$, constructed from the components of the motion tensor, where

$$\underline{\underline{T}}^\flat = \underline{\underline{I}} + \begin{bmatrix} \text{tr}(\underline{\underline{R}}^\flat) & 2\text{axial}^T(\underline{\underline{R}}^\flat) \\ 2\text{axial}(\underline{\underline{R}}^\flat) & 2\text{symm}(\underline{\underline{R}}^\flat) - \text{tr}(\underline{\underline{R}}^\flat)\underline{\underline{I}} \end{bmatrix}, \tag{42a}$$

$$\underline{\underline{T}}^\sharp = \begin{bmatrix} \text{tr}(\underline{\underline{R}}^\sharp) & 2\text{axial}^T(\underline{\underline{R}}^\sharp) \\ 2\text{axial}(\underline{\underline{R}}^\sharp) & 2\text{symm}(\underline{\underline{R}}^\sharp) - \text{tr}(\underline{\underline{R}}^\sharp)\underline{\underline{I}} \end{bmatrix}. \tag{42b}$$

Introducing eqs. (27) then yields

$$\underline{\underline{\mathcal{T}}} = 4\hat{e}\hat{e}^\dagger. \tag{43}$$

Two dual scalars, $\boldsymbol{\epsilon}_k$ and $\boldsymbol{\tau}_{k,\ell}$, are defined such that $\text{axial}(\boldsymbol{\epsilon}_k) = \langle e_k^\flat, e_k^\sharp \rangle$ and $\text{axial}(\boldsymbol{\tau}_{k,\ell}) = \langle T_{k,\ell}^\flat, T_{k,\ell}^\sharp \rangle$. Equation (43) implies $\boldsymbol{\tau}_{k,\ell} = 4\boldsymbol{\epsilon}_k\boldsymbol{\epsilon}_\ell$. Euler's parameters can be calculated as $2\boldsymbol{\epsilon}_m = \sqrt{\boldsymbol{\tau}_{m,m}}$ and $2\boldsymbol{\epsilon}_\ell = \boldsymbol{\tau}_{\ell,m}/\sqrt{\boldsymbol{\tau}_{m,m}}$. Because the evaluation of $\sqrt{\boldsymbol{\tau}_{m,m}}$ involves a division by $e_m^\flat$, the most accurate results will be obtained be selecting index m such that $T_{m,m}^\flat \geq T_{k,k}^\flat$ for $k = 1, 2, 3, 4$, leading to

$$\boldsymbol{\epsilon}_\ell = \frac{\boldsymbol{\tau}_{\ell,m}}{2\sqrt{\boldsymbol{\tau}_{m,m}}}. \tag{44}$$

The determination of quaternion $\hat{e}^\flat$ follows the algorithm developed decades ago by Klumpp [32] and Shepperd [33].

4.5 Discussion

This section has focused on the description of motion in terms of Euler motion parameters. The motion tensor, velocity vector, and the composition of motion formula were presented in sections 4.1, 4.2, and 4.3, respectively. The results obtained in this section were derived from the corresponding results for the geometric description of motion. A singularity free algorithm that extracts the motion parameters from the motion tensor is provided by eq. (44); this algorithm generalizes its counterpart for the extraction of Euler parameters from the rotation tensor. All operations for the manipulation of motion become quadratic when expressed in terms of Euler motion parameters.

5 Vectorial parameterization of motion

In section 3, motion was described geometrically based on Chasles' line and the magnitudes of the rotation and intrinsic displacement. On the other hand, a representation of motion based on eight parameters was presented in section 4, but the eight parameters are linked by the two constraints requiring the dual quaternion to be unit. In many cases, it is desirable to work with a parameterization of motion involving six parameters only.

Equation (24) introduced dual vector $\underline{e} = \sin \boldsymbol{\phi}/2 \ \bar{n}$ as the product of a dual scalar, $\sin \boldsymbol{\phi}/2$, by the Plücker coordinated of Chasles' line. Clearly, dual vector $\underline{e}$ gathers all the information about the motion: Chasles' line, $\bar{n}$, and the two scalars, $\text{axial}(\boldsymbol{\phi}) = \langle \phi, \delta \rangle$. The vectorial parameterization of motion [23] is more general and defined as

$$\underline{p} = \boldsymbol{\varpi}(\boldsymbol{\phi})\bar{n} = \frac{2}{\boldsymbol{\nu}}\bar{e}, \tag{45}$$

where $\boldsymbol{\varpi}$ is a dual function of dual scalar $\boldsymbol{\phi}$, called the "generating function." Because this function must be analytic, $\text{axial}(\boldsymbol{\varpi}(\boldsymbol{\phi})) = \langle \varpi(\phi), \varpi'(\phi)\delta \rangle$; $\varpi(\phi)$ is an arbitrary odd function of the magnitude of the rotation, ϕ, and notation $(\cdot)'$ indicates a derivative with respect to ϕ. Two dual functions play an important role in the vectorial parameterization of motion,

$$\boldsymbol{\nu} = \frac{2 \sin \boldsymbol{\phi}/2}{\boldsymbol{\varpi}}, \quad \boldsymbol{\varepsilon} = \frac{2 \tan \boldsymbol{\phi}/2}{\boldsymbol{\varpi}}. \tag{46}$$

By construction, dual vector $\underline{p}$ is not unit. Equation (20) implies $\underline{\underline{\mathcal{R}}}\,\underline{p} = \underline{p}$, *i.e.*, dual vector $\underline{p}$ is an eigenvector of the motion tensor associated with its unit eigenvalue.

5.1 The motion tensor

Introducing the vectorial parameterization of motion into eq. (20) yields the expression for the motion tensor,

$$\underline{\underline{\mathcal{R}}}(\underline{p}) = \underline{\underline{I}} + \boldsymbol{\zeta}_1 \tilde{p} + \boldsymbol{\zeta}_2 \tilde{p}\tilde{p}, \tag{47}$$

where

$$\boldsymbol{\zeta}_1(\boldsymbol{\phi}) = \frac{\sin \boldsymbol{\phi}}{\boldsymbol{\varpi}} = \frac{\boldsymbol{\nu}^2}{\boldsymbol{\varepsilon}}, \quad \boldsymbol{\zeta}_2(\boldsymbol{\phi}) = \frac{1 - \cos \boldsymbol{\phi}}{\boldsymbol{\varpi}^2} = \frac{\boldsymbol{\nu}^2}{2}. \tag{48}$$

5.2 The velocity vector

Taking a time derivative of the motion parameter vector yields $\dot{\underline{p}} = \boldsymbol{\varpi}'\dot{\boldsymbol{\phi}}\bar{n} + \boldsymbol{\varpi}\dot{\bar{n}}$, where $\boldsymbol{\varpi}' = \mathrm{d}\boldsymbol{\varpi}/\mathrm{d}\phi$. Identity (16b) leads to $\tilde{\bar{n}}\tilde{\bar{n}}\dot{\underline{p}} = \boldsymbol{\varpi}\tilde{\bar{n}}\tilde{\bar{n}}\dot{\bar{n}} = -\boldsymbol{\varpi}\dot{\bar{n}} = \boldsymbol{\varpi}'\dot{\boldsymbol{\phi}}\bar{n} - \dot{\underline{p}}$, because $\bar{n}$ is a unit dual vector.

Introducing these results into eqs. (33) then leads to

$$\underline{v} = \underline{\underline{\mathcal{H}}}(\underline{p})\dot{\underline{p}}, \qquad \underline{v}^* = \underline{\underline{\mathcal{H}}}^{\dagger}(\underline{p})\dot{\underline{p}}, \tag{49a}$$

$$\dot{\underline{p}} = \underline{\underline{\mathcal{H}}}^{-1}(\underline{p})\underline{v}, \quad \dot{\underline{p}} = \underline{\underline{\mathcal{H}}}^{-\dagger}(\underline{p})\underline{v}^*. \tag{49b}$$

Dual matrix $\underline{\underline{\mathcal{H}}}(\underline{p})$ and its inverse, referred as *tangent tensors*, are

$$\underline{\underline{\mathcal{H}}}(\underline{p}) = \boldsymbol{\sigma}_0\underline{\underline{I}} + \boldsymbol{\zeta}_2\tilde{p} + \boldsymbol{\sigma}_2\tilde{p}\tilde{p}, \tag{50a}$$

$$\underline{\underline{\mathcal{H}}}^{-1}(\underline{p}) = \boldsymbol{\chi}_0\underline{\underline{I}} - \frac{1}{2}\tilde{p} + \boldsymbol{\chi}_2\tilde{p}\tilde{p}. \tag{50b}$$

where the following dual functions were defined

$$\boldsymbol{\sigma}_0 = \frac{\boldsymbol{\iota}}{\boldsymbol{\varpi}'}, \quad \boldsymbol{\sigma}_2 = \frac{\boldsymbol{\sigma}_0 - \boldsymbol{\zeta}_1}{\boldsymbol{\varpi}^2}, \tag{51a}$$

$$\boldsymbol{\chi}_0 = \boldsymbol{\varpi}', \qquad \boldsymbol{\chi}_2 = \frac{1}{\boldsymbol{\varpi}^2}\left(\boldsymbol{\varpi}' - \frac{1}{\boldsymbol{\varepsilon}}\right). \tag{51b}$$

Dual matrix $\underline{\underline{\mathcal{H}}}$ enjoys the following remarkable properties,

$$\underline{\underline{\mathcal{H}}}(\underline{p}) = \underline{\underline{\mathcal{H}}}^{\dagger}(-\underline{p}), \tag{52a}$$

$$\underline{\underline{\mathcal{R}}} = \underline{\underline{\mathcal{H}}}\,\underline{\underline{\mathcal{H}}}^{-\dagger} = \underline{\underline{\mathcal{H}}}^{-\dagger}\underline{\underline{\mathcal{H}}}, \tag{52b}$$

$$\underline{\underline{\mathcal{R}}} - \underline{\underline{I}} = \tilde{p}\underline{\underline{\mathcal{H}}} = \underline{\underline{\mathcal{H}}}\,\tilde{p}, \tag{52c}$$

$$\underline{\underline{I}} - \underline{\underline{\mathcal{R}}}^{\dagger} = \boldsymbol{\nu}^2\tilde{p}\underline{\underline{\mathcal{H}}}^{-1} = \boldsymbol{\nu}^2\underline{\underline{\mathcal{H}}}^{-1}\tilde{p}, \tag{52d}$$

$$\tilde{p} = \underline{\underline{\mathcal{H}}}^{-\dagger} - \underline{\underline{\mathcal{H}}}^{-1}. \tag{52e}$$

5.3 Determination of the motion parameter vector

The determination of the vectorial parameterization from the motion tensor can be accomplished through a two step procedure: first, extract Euler parameters from the motion tensor using eq. (44), and second, express the vectorial parameterization in terms of Euler parameters using eq. (45).

5.4 Composition of motion

The concept of composition of motions was discussed in section 4.3. Let $\underline{p}_1$, $\underline{p}_2$, and $\underline{p}$ with parameters $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, and $\boldsymbol{\phi}$, respectively, be the motion parameter vectors of three motion tensors such that $\underline{\underline{\mathcal{R}}}(\underline{p}) = \underline{\underline{\mathcal{R}}}(\underline{p}_1)\underline{\underline{\mathcal{R}}}(\underline{p}_2)$. The relationship between the various parameters then follows from eq. (35)

$$\cos\frac{\boldsymbol{\phi}}{2} = \boldsymbol{\nu}_1\boldsymbol{\nu}_2\left(\frac{1}{\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2} - \frac{1}{4}\underline{p}_1^{\dagger}\underline{p}_2\right) = \frac{\boldsymbol{\nu}}{\boldsymbol{\varepsilon}}, \tag{53a}$$

$$\boldsymbol{\nu}\underline{p} = \boldsymbol{\nu}_1\boldsymbol{\nu}_2\left(\frac{1}{\boldsymbol{\varepsilon}_2}\underline{p}_1 + \frac{1}{\boldsymbol{\varepsilon}_1}\underline{p}_2 + \frac{1}{2}\tilde{p}_1\underline{p}_2\right). \tag{53b}$$

The first equation is used to compute $\boldsymbol{\phi}$ and hence, $\boldsymbol{\nu}$. The second equation then yields the components of the motion parameter vector.

5.5 The displacement vector

The vectorial parameterization of motion is based on dual vector $\underline{p}$ defined by eq. (45), whose primal and dual parts are related to the rotation and displacement components of the motion. Often, the displacement vector of a specific point of the body must be evaluated. Equation (23) provides the displacement vector, $\underline{u}$, of the point of rigid body that coincides with the origin of the reference frame, $\underline{u} = \delta\bar{n} + [\sin\phi + (1-\cos\phi)\tilde{n}]\underline{m}$, where $(\bar{n}, \underline{m})$ are the Plücker coordinates of Chales' line. Definition (45) of the motion parameter vector implies $\underline{m} = (\underline{p}^\sharp - \delta\varpi'\bar{n})/\varpi$ and hence, $\bar{n}^T\underline{p}^\sharp = \delta\varpi'$. With these results, the displacement vector becomes

$$\underline{u} = \underline{\underline{H}}^\flat \underline{p}^\sharp, \tag{54}$$

where $\underline{\underline{H}}^\flat$ is the primal part of the tangent tensor defined by eq. (50a).

5.6 Three specific parameterizations

This section focuses on three specific parameterizations of motion: the natural (or Cartesian), Cayley (or Cayley-Gibbs-Rodrigues) and Wiener-Milenković parameterizations. The first parameterization is obtained from the logarithmic map of the motion tensor, and the last two are derived from Cayley's first- and second-order decompositions. Note that dual vector $\underline{e}$ defined by eq. (24) also defines a parameterization of motion corresponding to generating function $\boldsymbol{\varpi}(\boldsymbol{\phi}) = \sin\boldsymbol{\phi}/2$.

5.6.1 The natural or Cartesian parameterization

The natural parameterization can be expressed as a logarithmic map of motion tensor, $\log(\underline{\underline{\mathcal{R}}}) = \log(\exp(\boldsymbol{\phi}\tilde{\underline{n}})) = \tilde{\underline{p}}$, where the first equality follows from eq. (21). This leads to the following parameterization of motion

$$\underline{p} = \boldsymbol{\phi}\bar{\underline{n}}, \tag{55}$$

which corresponds to generating function $\boldsymbol{\varpi} = \boldsymbol{\phi}$. Singularities will occur at $\phi = \pm\pi$ for the natural parameterization, because $\phi = \pm\pi$ correspond to the same rotation (or motion) and hence, it cannot represent rotations of arbitrary magnitude.

The generating function is obtained easily as $\boldsymbol{\phi} = (\underline{p}^\dagger\underline{p})^{1/2}$ and eq. (46) yields $\boldsymbol{\nu} = \sin\boldsymbol{\phi}/2/(\boldsymbol{\phi}/2)$ and $\boldsymbol{\varepsilon} = \tan\boldsymbol{\phi}/2/(\boldsymbol{\phi}/2)$. The motion tensor now follows from eqs. (47) and (48) as

$$\underline{\underline{\mathcal{R}}} = \exp(\boldsymbol{\phi}\tilde{\underline{n}}) = \underline{\underline{I}} + \frac{\sin\boldsymbol{\phi}}{\boldsymbol{\phi}}\tilde{\underline{p}} + \frac{\boldsymbol{\iota} - \cos\boldsymbol{\phi}}{\boldsymbol{\phi}^2}\tilde{\underline{p}}\tilde{\underline{p}}. \tag{56}$$

Similarly, the tangent tensor and its inverse are obtained from eqs. (50a) and (51) as

$$\underline{\underline{\mathcal{H}}} = \underline{\underline{I}} + \frac{\boldsymbol{\iota} - \cos\boldsymbol{\phi}}{\boldsymbol{\phi}^2}\tilde{\underline{p}} + \frac{\boldsymbol{\iota} - (\sin\boldsymbol{\phi})/\boldsymbol{\phi}}{\boldsymbol{\phi}^2}\tilde{\underline{p}}\tilde{\underline{p}}, \tag{57a}$$

$$\underline{\underline{\mathcal{H}}}^{-1} = \underline{\underline{I}} - \frac{1}{2}\tilde{\underline{p}} + \frac{\boldsymbol{\iota} - (\boldsymbol{\phi}/2)/\tan\boldsymbol{\phi}/2}{\boldsymbol{\phi}^2}\tilde{\underline{p}}\tilde{\underline{p}}. \tag{57b}$$

For the Cartesian parameterization of motion, the composition of motion formulæ (53) cannot be obtained in closed form.

5.6.2 The first-order Cayley or Cayley-Gibbs-Rodrigues parameterization

The first-order Cayley decomposition given by eq. (29) suggests the following parameterization of motion

$$\underline{c} = 2\bar{\underline{n}}\tan\frac{\boldsymbol{\phi}}{2} = \frac{2}{\boldsymbol{\mu}}\underline{e}, \tag{58}$$

which corresponds to generating function $\boldsymbol{\varpi} = 2\tan\boldsymbol{\phi}/2$. A factor of two was added as is customary in the literature; it enforces the condition $\lim_{\boldsymbol{\phi}\to\mathbf{o}} = \boldsymbol{\phi}\bar{n}$, *i.e.*, the infinitesimal motion vector is recovered for small motions. Clearly, this parameterization is singular when $\phi \to \pm\pi$ and hence, it cannot represent rotations of arbitrary magnitude.

The following parameter is introduced: $\boldsymbol{\alpha} = \boldsymbol{\mu}^2 = \boldsymbol{\iota}/(\boldsymbol{\iota} + \underline{c}^\dagger\underline{c}/4)$. Equation (46) yields $\boldsymbol{\nu} = \cos\boldsymbol{\phi}/2 = \boldsymbol{\mu} = \sqrt{\boldsymbol{\alpha}}$ and $\boldsymbol{\varepsilon} = \boldsymbol{\iota}$. The motion tensor now follows from eqs. (47) and (48) as

$$\underline{\underline{\mathcal{R}}} = \underline{\underline{I}} + \boldsymbol{\alpha}\tilde{c} + \frac{\boldsymbol{\alpha}}{2}\tilde{c}\tilde{c}. \tag{59}$$

Similarly, the tangent tensor and its inverse are obtained from eqs. (50a) and (51) as

$$\underline{\underline{\mathcal{H}}} = \boldsymbol{\alpha} + \frac{\boldsymbol{\alpha}}{2}\tilde{c}, \tag{60a}$$

$$\underline{\underline{\mathcal{H}}}^{-1} = \frac{\boldsymbol{\iota}}{\boldsymbol{\alpha}} - \frac{1}{2}\tilde{c} + \frac{1}{4}\tilde{c}\tilde{c}. \tag{60b}$$

For the Cayley-Gibbs-Rodrigues parameterization, the composition of motion formula (53) takes a particularly simple form

$$\underline{c} = \frac{\boldsymbol{\iota}}{\boldsymbol{\iota} - \underline{c}_1^\dagger\underline{c}_2/4}(\underline{c}_1 + \underline{c}_2 + \frac{1}{2}\tilde{c}_1\underline{c}_2). \tag{61}$$

5.6.3 The second-order Cayley or Wiener-Milenković parameterization

The second-order Cayley decomposition given by eq. (29) suggests the following parameterization of motion

$$\underline{w} = 4\bar{n}\tan\frac{\boldsymbol{\phi}}{4} = \frac{4}{\boldsymbol{\iota} + \boldsymbol{\mu}}\underline{e}, \tag{62}$$

which corresponds to generating function $\boldsymbol{\varpi} = 4\tan\boldsymbol{\phi}/4$. A factor of four was added as is customary in the literature; it enforces the condition $\lim_{\boldsymbol{\phi}\to\mathbf{o}} = \boldsymbol{\phi}\bar{n}$, *i.e.*, the infinitesimal motion vector is recovered for small motions. From the second equality of eq. (62), the second-order Cayley parameterization appears to be a stereographic projection of the dual unit quaternions $\hat{e}$, as discussed by Hurtado [34]. The singularity of this parameterization now occurs for $\phi = \pm 2\pi$ and hence, within the range $\phi \in [-\pi, \pi]$ it can handle all rigid-body motions.

The following parameters are introduced: $\boldsymbol{\nu} = \cos^2\boldsymbol{\phi}/4 = \boldsymbol{\iota}/(\boldsymbol{\iota} + \underline{w}^\dagger\underline{w}/16)$ and $\boldsymbol{\varepsilon} = \boldsymbol{\iota}/(\boldsymbol{\iota} - \underline{w}^\dagger\underline{w}/16)$; note that $2\boldsymbol{\nu}\boldsymbol{\varepsilon} = \boldsymbol{\nu} + \boldsymbol{\varepsilon}$. The motion tensor now follows from eqs. (47) and (48) as

$$\underline{\underline{\mathcal{R}}} = \underline{\underline{I}} + \frac{\boldsymbol{\nu}^2}{\boldsymbol{\varepsilon}}\tilde{w} + \frac{\boldsymbol{\nu}^2}{2}\tilde{w}\tilde{w}. \tag{63}$$

Similarly, the tangent tensor and its inverse are obtained from eqs. (50a) and (51) as

$$\underline{\underline{\mathcal{H}}} = \boldsymbol{\nu}(\underline{\underline{I}} + \frac{\boldsymbol{\nu}}{2}\tilde{w} + \frac{\boldsymbol{\nu}}{8}\tilde{w}\tilde{w}), \tag{64a}$$

$$\underline{\underline{\mathcal{H}}}^{-1} = \frac{\boldsymbol{\iota}}{\boldsymbol{\nu}} - \frac{1}{2}\tilde{w} + \frac{1}{8}\tilde{w}\tilde{w}. \tag{64b}$$

Because motion tensors $\underline{\underline{\mathcal{R}}}(\boldsymbol{\phi}, \bar{n})$ and $\underline{\underline{\mathcal{R}}}(\boldsymbol{\phi}\pm 2\pi, \bar{n})$ describe the same geometric rigid-body motion, a second set of parameters, referred to as the *shadow* parameters [34], are introduced,

$$\underline{w}^* = 4\tan\frac{\boldsymbol{\phi}\pm 2\pi}{4}\bar{n} = -4\cot\frac{\boldsymbol{\phi}}{4}\bar{n} = -\frac{\underline{w}}{\tan^2\boldsymbol{\phi}/4} = -\frac{\boldsymbol{\nu}}{\boldsymbol{\iota} - \boldsymbol{\nu}}\underline{w}. \tag{65}$$

It now follows $\|\underline{w}^*\| = \|\underline{w}\|/\tan^2\boldsymbol{\phi}/4$ and hence, $\|\underline{w}^*\|\|\underline{w}\| = 16$. Because $\nu^\flat = \cos^2\phi/4$, $\nu^\flat \leq 1/2$ for $\pi \leq |\phi| \leq 2\pi$ whereas $\nu^\flat \geq 1/2$ for $|\phi| \leq \pi$. To avoid singularities when $|\phi| \leq \pi$, eq. (65)

is used to compute the shadow parameters, which can be interpreted as a rescaling operation $\|\underline{w}^*\|\|\underline{w}\| = 16$.

The rescaling and composition of motion operations are combined in to a single operation easily. First, eq. (53a) provides parameter $\boldsymbol{\nu}$ of the composed motion as $\boldsymbol{\nu}/\boldsymbol{\varepsilon} = 2\boldsymbol{\nu} - \boldsymbol{\iota} = \boldsymbol{\nu}_1\boldsymbol{\nu}_2(\boldsymbol{\iota}/\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2 - \underline{p}_1^\dagger\underline{p}_2/4)$. The composed motion now follows from eq. (53b) as

$$\underline{w} = \boldsymbol{\nu}_1\boldsymbol{\nu}_2(\frac{\underline{w}_1}{\boldsymbol{\varepsilon}_2} + \frac{\underline{w}_2}{\boldsymbol{\varepsilon}_1} + \frac{1}{2}\tilde{\underline{w}}_1\underline{w}_2)\begin{cases} \boldsymbol{\iota}/\boldsymbol{\nu}, & \nu^\flat \geq 1/2, \\ -\boldsymbol{\iota}/(\boldsymbol{\iota} - \boldsymbol{\nu}), & \nu^\flat \leq 1/2. \end{cases} \tag{66}$$

5.7 Discussion

This section has presented the vectorial parameterization of motion. The motion tensor, velocity vector, and the composition of motion formula were presented in sections 5.1, 5.2, and 5.4, respectively. Here again, the results were derived from the corresponding results for the geometric description of motion. The vectorial parameterization of motion is, in fact, a family of parameterizations, each one associated with a generating function, $\varpi(\phi)$.

The naming used for the parameterizations presented above is inherited from the naming of the corresponding parameterization of rotation. For instance, Wiener [35] and Milenković [36] presented the parameterization of rotation based on generating function $\varpi^\flat = 4\tan\boldsymbol{\phi}/4$. Generating function $\boldsymbol{\varpi} = 4\tan\boldsymbol{\phi}/4$, implying $\varpi^\sharp = \delta/\cos^2\boldsymbol{\phi}/4$, leads to the "Wiener-Milenković" parameterization of motion. The use of dual functions renders this transition seamless and eases the associated algebra considerably: rotation and motion parameterizations are treated simultaneously.

An attractive feature of the Cayley-Gibbs-Rodrigues and Wiener-Milenković parameterizations of rotation, is that all trigonometric functions have been eliminated: they are purely algebraic parameterizations of rotation. This feature remains true for the correspond parameterizations of motion, lowering the computational cost associated with motion operations.

6 Interpolation of motion

In many applications, interpolation of motion is required. For instance, in the finite element method, the motion field and its derivative must be interpolated over one element to evaluate the strain field. In robotic applications, path planning often involves the interpolation of motion.

Consider a motion defined by its invariant Chasles' line, $\bar{n}$, and its scalar characteristics, $\boldsymbol{\phi} = \boldsymbol{\Omega}s$. If parameter s is interpreted as time, eq (33a) yield a constant velocity $\underline{v} = \dot{\boldsymbol{\phi}}\bar{n} = \boldsymbol{\Omega}\bar{n}$, because Chasles' line is invariant and hence, $\dot{\bar{n}} = \underline{0}$. The motion represents the trajectory of a rigid body moving at a constant velocity along a helicoidal trajectory. If parameter s is interpreted as space, the same results hold, but $\boldsymbol{\Omega}$ now represents the constant curvature of the path. For instance, the motion now represents the configuration of the cross-section of a naturally curved and twisted beam; the axis of the beam is a helix and $\boldsymbol{\Omega}$ its constant curvature.

To simplify the development, the problem is formulated in the motion's canonical frame, which implies $\bar{n} = \bar{\imath}_3$. Three frames, corresponding to $s = 0, s$, and 1, are defined, as $\mathcal{F}_1$, $\mathcal{F}$, and $\mathcal{F}_2$, respectively. It is verified easily that the Euler parameters of the three frames, denoted $\hat{e}_1$, $\hat{e}$, and $\hat{e}_2$, respectively, are

$$\hat{e}_1^\flat = \begin{Bmatrix} 1 \\ 0\ \bar{\imath}_3 \end{Bmatrix}, \quad \hat{e}^\flat = \begin{Bmatrix} \cos(\Omega s/2) \\ \sin(\Omega s/2)\ \bar{\imath}_3 \end{Bmatrix}, \quad \hat{e}_2^\flat = \begin{Bmatrix} \cos(\Omega/2) \\ \sin(\Omega/2)\ \bar{\imath}_3 \end{Bmatrix}, \tag{67a}$$

$$\hat{e}_1^\sharp = \begin{Bmatrix} 0 \\ 0\ \bar{\imath}_3 \end{Bmatrix}, \quad \hat{e}^\sharp = \frac{\Delta s}{2}\begin{Bmatrix} -\sin(\Omega s/2) \\ \cos(\Omega s/2)\ \bar{\imath}_3 \end{Bmatrix}, \quad \hat{e}_2^\sharp = \frac{\Delta}{2}\begin{Bmatrix} -\sin(\Omega/2) \\ \cos(\Omega/2)\ \bar{\imath}_3 \end{Bmatrix}, \tag{67b}$$

where $\text{axial}(\boldsymbol{\Omega}) = \langle \Omega, \Delta \rangle$. Tedious algebra involving the use of common trigonometric identities reveals that $\sin(\boldsymbol{\Omega}/2)\ \hat{e} = \sin((1-s)\boldsymbol{\Omega}/2)\ \hat{e}_1 + \sin(s\boldsymbol{\Omega}/2)\ \hat{e}_2$. For convenience, the following change of variable is performed, $s = (1+\xi)/2$, and the interpolation formula becomes

$$\hat{e}(\xi) = \frac{\sin h_1(\xi)\boldsymbol{\Omega}/2}{\sin \boldsymbol{\Omega}/2}\hat{e}_1 + \frac{\sin h_2(\xi)\boldsymbol{\Omega}/2}{\sin \boldsymbol{\Omega}/2}\hat{e}_2, \tag{68}$$

where $h_1(\xi) = (1-\xi)/2$ and $h_2(\xi) = (1+\xi)/2$ are the shape functions of the interpolation, which have been cast in the standard form used in finite element formulations. Because the motion tensor commutes with dual scalars, eq. (68) remains true when the components of dual quaternions $\hat{e}$, $\hat{e}_1$ and $\hat{e}_2$ are resolved in any common frame, *i.e.*, the interpolation is frame invariant.

The notation adopted in this paper shows that the following interpolation formula holds for the rotation part of the dual quaternions, $\sin(\Omega/2)\ \hat{e}^\flat(\xi) = \sin(h_1\Omega/2)\hat{e}_1^\flat + \sin(h_2\Omega/2)\hat{e}_2^\flat$. This expression corresponds to the *spherical linear interpolation*, abbreviated as "Slerp," proposed by Shoemake [37] for computer animation applications as a constant velocity interpolation.

If the curvature is small, *i.e.*, $\boldsymbol{\Omega} \to \mathbf{o}$ the shape functions simplify considerably: $\lim_{\boldsymbol{\Omega}\to\mathbf{o}}(\sin h_i\boldsymbol{\Omega}/2)/(\sin \boldsymbol{\Omega}/2) \approx h_i$. The motion interpolation formula then becomes

$$\hat{e}(\xi) \approx h_1(\xi)\hat{e}_1 + h_2(\xi)\hat{e}_2. \tag{69}$$

Clearly, when curvatures are small, the use of simple polynomial shape functions yields a nearly constant curvature interpolation. Equation (69), however, is not equivalent to the traditional interpolation used in the finite element method. Rather than interpolating displacement and rotation fields separately, as is done commonly in the finite element method, the Euler motion parameters are interpolated here.

When cast in the form of eq. (68), the constant curvature interpolation is similar to classical polynomial interpolation formulæ: the interpolated motion, $\hat{e}(\xi)$, is a combination of the motions at the end nodes, $\hat{e}_1$ and $\hat{e}_2$. But important differences exist. First, for this two node interpolation problem, dual shape functions replace the classical shape functions. Second, these dual shape functions are transcendental functions rather than the polynomial functions used in the classical approach. Finally, the shape functions depend on the relative motion of the nodes: the interpolation is inherently nonlinear.

Although interpolation scheme (68) is more complex than its polynomial counterpart used in the finite element method, it presents three important properties: *(1)* it can represent constant curvature states exactly, *(2)* it holds when resolved in any frame, and *(3)* because it depends on the relative motion between the end points only, the interpolation remains invariant under composition of a rigid-body motion, *i.e.*, the interpolation is objective.

7 Conclusion

This paper has presented a comprehensive treatment of the representation and manipulation of motion using a notation that eliminates the bookkeeping parameter typically used in dual number algebra, thereby recasting all operations within the framework of linear algebra and streamlining the process. Expressions were presented for the motion tensor, velocity vector, and composition of motion when using the geometric description of motion, Euler parameters, and the vectorial parameterization of motion. All developments were presented within the framework of dual numbers directly; the principle of transference was not invoked: the manipulation of rotation was found to be a particular case of that of motion, as should be.

While the concept of dual algebra is not new, the formalism presented in this paper should promote its acceptance outside of the fields of kinematics and robotics. Indeed, the proposed approach is based on the well-known rules of linear algebra, which researchers in the field of multibody dynamics are well versed in.

No attempt was made to prove the principle of transference, which was not used at any point in the developments. The paper, however, presents numerous illustration of this principle that is embedded in the proposed notation.

Application of dual algebra to the vectorial parameterization of motion eases the associated algebra considerably. Because all dual functions are selected to be analytic, the generating functions used in the parametrization of rotation generalize to dual generating functions immediately. All the formulæ for manipulating motion then follow from their counterpart for the geometric description of motion. Because it is a purely kinematic problem, the interpolation of motion benefits from a formulation within the framework of dual algebra. The shape functions become dual functions and their derivation is straightforward.

A Dual functions

Functions of dual variables play an important role in the developments presented in this paper. It was established that functions of dual variable must be of the form given by eq. (13) and hence, trigonometric functions of dual scalars present the following form

$$\sin \boldsymbol{\phi} = \begin{bmatrix} \sin \phi^\flat & \phi^\sharp \cos \phi^\flat \\ 0 & \sin \phi^\flat \end{bmatrix}, \ \cos \boldsymbol{\phi} = \begin{bmatrix} \cos \phi^\flat & -\phi^\sharp \sin \phi^\flat \\ 0 & \cos \phi^\flat \end{bmatrix}, \ \tan \boldsymbol{\phi} = \begin{bmatrix} \tan \phi^\flat & \phi^\sharp / \cos^2 \phi^\flat \\ 0 & \tan \phi^\flat \end{bmatrix} \tag{70}$$

The same result can the obtained by expanding the sine or cosine function in Taylor series,

$$\sin \boldsymbol{\phi} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \boldsymbol{\phi}^{2k+1}, \quad \cos \boldsymbol{\phi} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \boldsymbol{\phi}^{2k}, \tag{71}$$

and using eq. (5) to evaluate the powers of the dual variable.

Appropriate care must be taken when dealing with dual functions of several variables. For instance, taking into account the general form of a dual function given by eq. (13), the product of two dual functions become

$$\boldsymbol{\theta}(\boldsymbol{\alpha})\boldsymbol{\lambda}(\boldsymbol{\beta}) = \begin{bmatrix} \theta^\flat \lambda^\flat & \alpha^\sharp \dfrac{\mathrm{d}(\theta^\flat \lambda^\flat)}{\mathrm{d}\alpha^\flat} + \beta^\sharp \dfrac{\mathrm{d}(\theta^\flat \lambda^\flat)}{\mathrm{d}\beta^\flat} \\ 0 & \theta^\flat \lambda^\flat \end{bmatrix}, \tag{72}$$

where $\theta^\flat = \theta^\flat(\alpha^\flat)$ and $\lambda^\flat = \lambda^\flat(\beta^\flat)$. With this result at hand, it follows

$$\sin(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \begin{bmatrix} \sin(\alpha^\flat + \beta^\flat) & (\alpha^\sharp + \beta^\sharp) \cos(\alpha^\flat + \beta^\flat) \\ 0 & \sin(\alpha^\flat + \beta^\flat) \end{bmatrix} = \sin \boldsymbol{\alpha} \cos \boldsymbol{\beta} + \cos \boldsymbol{\alpha} \sin \boldsymbol{\beta}. \tag{73}$$

Clearly, all the well-known trigonometric identities generalize to dual trigonometric functions. A second example of application of eq. (72) is

$$\frac{\sin \boldsymbol{\alpha}}{\sin \boldsymbol{\beta}} = \begin{bmatrix} \dfrac{\sin \alpha^\flat}{\sin \beta^\flat} & \alpha^\sharp \dfrac{\mathrm{d}}{\mathrm{d}\alpha^\flat} \left(\dfrac{\sin \alpha^\flat}{\sin \beta^\flat} \right) + \beta^\sharp \dfrac{\mathrm{d}}{\mathrm{d}\beta^\flat} \left(\dfrac{\sin \alpha^\flat}{\sin \beta^\flat} \right) \\ 0 & \dfrac{\sin \alpha^\flat}{\sin \beta^\flat} \end{bmatrix}, \tag{74}$$

which can be used to evaluate the dual functions appearing in eq. (68).

References

[1] W.K. Clifford. Preliminary sketch of biquaternions. *Proceedings of the London Mathematical Society*, s1-4(1):381–395, 1871.

[2] E. Study. *Geometrie der Dynamen*. Teubner, Leipzig, 1903.

[3] J. Rooney. On the principle of transference. In *Proceedings of the IV IFToMM Congress*, pages 1089–1094, New Castle Upon Tyne, UK, 1975.

[4] J.M.R. Martinez and J. Duffy. The principle of transference: History, statement and proof. *Mechanism and Machine Theory*, 28(1):165–177, 1993.

[5] A.T. Yang and F. Freudenstein. Application of dual-number quaternion algebra to the analysis of spatial mechanims. *ASME Journal of Applied Mechanics*, 86:300–308, 1964.

[6] F.M. Dimentberg. The screw calculus and its applications. Technical Report AD 680993, Clearinghouse for Federal and Scientific Technical Information, Virginia, USA, April 1968.

[7] O. Bottema and B. Roth. *Theoretical Kinematics*. Dover Publications, Inc., New York, 1979.

[8] J.M. McCarthy. *An Introduction to Theoretical Kinematics*. The MIT Press, Cambridge, MA, 1990.

[9] J. Angeles. The application of dual algebra to kinematic analysis. In J. Angeles and E. Zakhariev, editors, *Computational Methods in Mechanical Systems*, volume 161, pages 3–31. Springer-Verlag, Heidelberg, 1998.

[10] E. Pennestrì and R. Stefanelli. Linear algebra and numerical algorithms using dual numbers. *Multibody System Dynamics*, 18:323–344, 2007.

[11] M.L. Keler. Kinematics and statics including friction in single-loop mechanisms by screw calculus and dual vectors. *Journal of Engineering for Industry*, 95(2):471–480, 1973.

[12] V. Brodsky and M. Shoham. The dual inertia operator and its application to robot dynamics. *ASME Journal of Mechanical Design*, 116:1089–1095, 1994.

[13] V. Brodsky and M. Shoham. Dual numbers representation of rigid body dynamics. *Mechanism and Machine Theory*, 34:975–991, 1999.

[14] I.S. Fischer. *Dual Number Methods in Kinematics, Statics and Dynamics*. CRC Press, Boca Raton, 1999.

[15] O. Brüls and A. Cardona. On the use of Lie group time integrators in multibody dynamics. *Journal of Computational and Nonlinear Dynamics*, 5(3):031002 1–13, July 2010.

[16] Z. Terze, A. Müller, and D. Zlatar. Lie-group integration method for constrained multibody systems in state space. *Multibody System Dynamics*, 34(3):275–305, 2015.

[17] J.C. Simo. A finite strain beam formulation. The three-dimensional dynamic problem. Part I. *Computer Methods in Applied Mechanics and Engineering*, 49(1):55–70, 1985.

[18] V. Sonneville, A. Cardona, and O. Brüls. Geometrically exact beam finite element formulated on the special Euclidean group SE(3). *Computer Methods in Applied Mechanics and Engineering*, 268(1):451–474, 2014.

[19] F. Demoures, F. Gay-Balmaz, M. Kobilarov, and T.S. Ratiu. Multisymplectic Lie group variational integrator for a geometrically exact beam in R^3. *Communications in Nonlinear Science and Numerical Simulation*, 19(10):3492–3512, 2014.

[20] T. Merlini and M. Morandini. The helicoidal modeling in computational finite elasticity. Part I: Variational formulation. *International Journal of Solids and Structures*, 41(18-19):5351–5381, 2004.

[21] T. Merlini and M. Morandini. The helicoidal modeling in computational finite elasticity. Part II: Multiplicative interpolation. *International Journal of Solids and Structures*, 41(18-19):5383–5409, 2004.

[22] M. Borri, C.L. Bottasso, and L. Trainelli. A novel momentum-preserving energy-decaying algorithm for finite element multibody procedures. In Jorge Ambrósio and Michal Kleiber, editors, *Proceedings of Computational Aspects of Nonlinear Structural Systems with Large Rigid Body Motion, NATO Advanced Research Workshop, Pultusk, Poland, July 2-7*, pages 549–568, 2000.

[23] O.A. Bauchau and J.Y. Choi. The vector parameterization of motion. *Nonlinear Dynamics*, 33(2):165–188, 2003.

[24] M.A. Crisfield and G. Jelenić. Objectivity of strain measures in the geometrically exact three-dimensional beam theory and its finite-element implementation. *Proceedings of the Royal Society, London: Mathematical, Physical and Engineering Sciences*, 455(1983):1125–1147, 1999.

[25] O.A. Bauchau and S.L. Han. Interpolation of rotation and motion. *Multibody System Dynamics*, 31(3):339–370, 2014.

[26] W.R. Hamilton. *Elements of Quaternions*. Cambridge University Press, Cambridge, 1899.

[27] O.A. Bauchau. *Flexible Multibody Dynamics*. Springer, Dordrecht, Heidelberg, London, New-York, 2011.

[28] F.B. Hildebrand. *Advanced Calculus for Applications*. Prentice Hall, Inc., Englewood Cliffs, New Jersey, second edition, 1976.

[29] M. Chasles. Note sur les propriétés générales du système de deux corps semblables entre eux et placés d'une manière quelconque dans l'espace; et sur le déplacement fini, ou infiniment petit d'un corps solide libre. *Bulletin des Sciences Mathématiques de Férussac*, 14:321–326, 1830.

[30] H. Schaub and J.L. Junkins. Stereographic orientation parameters for attitude dynamics: A generalization of the rodrigues parameters. *The Journal of the Astronautical Sciences*, 44:1–19, 1996.

[31] P. Tsiotras, J.L. Junkins, and H. Schaub. Higher-order Cayley transforms with applications to attitude representation. *Journal of Guidance, Control, and Dynamics*, 20(3):528–534, May-June 1997.

[32] A.R. Klumpp. Singularity-free extraction of a quaternion from a direction-cosine matrix. *Journal of Spacecraft and Rockets*, 13:754–755, December 1976.

[33] S.W. Shepperd. Quaternion from rotation matrix. *Journal of Guidance and Control*, 1:223–224, May-June 1978.

[34] J.E. Hurtado. Interior parameters, exterior parameters, and a Cayley-like transform. *Journal of Guidance, Control, and Dynamics*, 32(2):653–657, April 2009.

[35] T.F. Wiener. *Theoretical Analysis of Gimballess Inertial Reference Equipment Using Delta-Modulated Instruments*. PhD thesis, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1962. Department of Aeronautical and Astronautical Engineering.

[36] V. Milenković. Coordinates suitable for angular motion synthesis in robots. In *Proceedings of the Robot VI Conference, Detroit MI, March 2-4, 1982*, 1982. Paper MS82-217.

[37] K. Shoemake. Animating rotation with quaternion curves. *SIGGRAPH Computer Graphics*, 19(3):245–254, 1985.